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ON THERMAL EFFECTS IN THE THEORY OF RODS

A. E. GREEN
Mathematical Institute, Oxford OX1 3LB, England

and

P. M. NAGHDI
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, U.S.A.

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Abstract—This paper is concerned with thermomechanics of slender rods by a direct approach based on the theory of a Cosserat curve comprising a one-dimensional curve and a pair of directors attached to every point of the curve. In all previous developments of the thermo-mechanical theory of rods by direct approach, only one temperature field has been admitted. This allows for the characterization of temperature changes along some reference curve, such as the line of centroids of the (three-dimensional) rod-like body, but not for temperature changes across the rod cross section. A main purpose of the present study is to incorporate the latter effect into the theory; and, in the context of the theory of a Cosserat curve, this is achieved by a recent approach of Green and Naghdi[1, 2] to thermomechanics which provides a natural way of introducing more than one temperature field at each material point of the curve. Apart from full discussion of thermomechanics of rods and thermodynamical restrictions arising from the second law of thermodynamics for rods, attention is given to a discussion of symmetries (including material symmetries) of rods which in a reference configuration are straight. The paper also contains a detailed discussion of the linear theory of straight, elastic, orthotropic rods, including the determination of the relevant constitutive coefficients.

1. INTRODUCTION

This paper is concerned with thermomechanics of slender rods by a direct approach based on the theory of *Cosserat* (or *directed*) *curves*. A Cosserat curve considered here is a body $\mathcal{R}$ comprising a one-dimensional curve (embedded in a Euclidean 3-space) and two directors (i.e. deformable vectors) attached to every point of the curve.† The development of a complete theory of a Cosserat or a directed curve with two directors begins with a paper of Green and Laws[3] whose derivation is carried out mainly from an appropriate energy equation, together with invariance requirements under superposed rigid body motions. A related theory of a directed curve with three deformable directors at each point of the curve, developed in the context of a purely mechanical theory and with the use of a virtual work principle, is given by Cohen[4]. A further development of the basic theory of a Cosserat curve along with certain general developments regarding the nonlinear and linear constitutive equations for elastic rods is contained in the more recent work of Green *et al.*[5]. For clarity's sake, we may recall that the material curve of $\mathcal{R}$ can be identified with a particular reference curve (often taken to be an interior curve) in the three-dimensional rod-like body, e.g. the line of centroids of the cross section of the rod in some fixed reference configuration; the directors at each point are regarded as representing the material filaments across the reference curve, i.e. in the cross section of the rod.

Throughout the previous developments of the thermo-mechanical theory of rods by direct approach, only one temperature field has been admitted and this allows for the characterization of temperature changes along the reference curve of the rod-like body. Some indication of how temperature changes *across* the reference curve of the rod-like body could be dealt with has been given in the paper of Green and Naghdi[6] by using three-dimensional approximations, but no direct thermo-mechanical theory of rods with more than one temperature has been discussed in the literature so far.

Although widespread use of the Clausius–Duhem inequalities has been made in three, two and one-dimensional continuum thermodynamics, these inequalities have been subject to the

†The body $\mathcal{R}$ is taken to model some of the properties of a three-dimensional body of rod-like character. When the directors are absent it reflects the properties of a material curve appropriate for the construction of string theory.

criticism that in some circumstances they do not reflect adequately ideas associated with the Second Law of Thermodynamics. Green and Naghdi[1] have developed a new approach to three-dimensional continuum thermomechanics which is independent of any particular mathematical expression of the second law and which imposes some restrictions on the constitutive assumptions leading to a reduction of a number of independent response functions (or functionals) in the set of constitutive assumptions. In the present paper the same approach is used for a Cosserat curve and this provides a natural way of introducing more than one temperature field.† When the directors are absent, the theory reduces to that of a material curve which may be a material curve surrounded by another continuum.

Specifically, the contents of the paper are as follows. Section 2 contains a concise summary of the various basic results of the purely mechanical theory of a Cosserat curve with two directors. With reference to thermal properties, in Section 3 we admit at each material point of the curve of $\mathcal{R}$ a number of different one-dimensional temperatures and different one-dimensional entropies, as well as related thermal fields; and, in parallel with one-dimensional conservation laws for balances of mass and momenta, we postulate balances of entropy. Next, we recall the balance of energy for the Cosserat curve; and, following the recent approach of Green and Naghdi[1], after elimination of the assigned fields—i.e. assigned director forces and external rates of supply of entropy—regard the resulting equation as an identity to be satisfied for all thermo-mechanical processes. In Section 4, we briefly discuss thermoelastic theory of a Cosserat curve on the basis of the new procedure in thermomechanics (see Section 3) and also compare the results with earlier developments (see Green and Laws[3], Green *et al.*[5]) involving only a single temperature.

A new inequality representing the second law of thermodynamics for rods based on the present authors' earlier work (Green and Naghdi[1, 2]), along with restrictions on heat flux vectors and the specific internal energy are obtained in Sections 5 and 8, respectively, while Sections 6 and 7 contain a discussion of relevant results for rods obtained from the three-dimensional theory. The last two sections, namely Sections 9 and 10, are devoted to a discussion of symmetries (including material symmetries) for rods and the linear thermoelastic theory of straight orthotropic rods. The developments in Sections 9 and 10 supplement our earlier results by direct approach (Green *et al.*[7]) for thermoelastic rods in the presence of a single temperature and previous values for constitutive coefficients in the linear theory.

2. SUMMARY OF MECHANICAL THEORY

In this section we summarize the main kinematics and the basic equations of the mechanical theory of rods based on the work of Green and Laws[3] in the form developed by Green, *et al.*[5]. A rod is a body $\mathcal{R}$ comprising a material curve with two deformable directors attached to every point of the curve. Let the particles of the material curve of $\mathcal{R}$ be identified with a convected coordinate ζ and let the material curve in the present configuration at time t be referred to as c. Let $\mathbf{r}$ be the position vector of c and $\mathbf{d}_\alpha$ ($\alpha = 1, 2$) the directors at $\mathbf{r}$. A motion of the rod is then defined by‡

$$\mathbf{r} = \mathbf{r}(\zeta, t), \qquad \mathbf{d}_\alpha = \mathbf{d}_\alpha(\zeta, t), \qquad [\mathbf{d}_1\mathbf{d}_2\mathbf{a}_3] > 0, \tag{2.1}$$

where

$$\mathbf{a}_3 = \mathbf{a}_3(\zeta, t) = \partial\mathbf{r}/\partial\zeta \tag{2.2}$$

is a vector tangent to the curve c and the directors $\mathbf{d}_\alpha$ have the property that they remain unaltered in magnitude under superposed rigid body motions. The velocity and director velocities are given by

$$\mathbf{v} = \dot{\mathbf{r}}(\zeta, t), \qquad \mathbf{w}_\alpha = \dot{\mathbf{d}}_\alpha(\zeta, t), \tag{2.3}$$

where a superposed dot stands for material time derivative with respect to t holding ζ fixed. In

†For the purely mechanical theory, it is already clear how to extend the theory with more than two directors.

‡The positive sign in $(2.1)_3$ is taken for definiteness. Alternatively, it will suffice to assume that $[\mathbf{d}_1\mathbf{d}_2\mathbf{a}_3] \neq 0$ with the understanding that in any given motion the scalar triple product $[\mathbf{d}_1\mathbf{d}_2\mathbf{a}_3]$ is either > 0 or < 0.

the reference configuration of $\mathscr{R}$ which we take to be the initial configuration, let the material curve of $\mathscr{R}$ be referred to by C and denote the initial position vector by $\mathbf{R}$, the tangent vector to C by $\mathbf{A}_3$ and the initial directors by $\mathbf{D}_\alpha$. Then,

$$\mathbf{R} = \mathbf{R}(\zeta) = \mathbf{r}(\zeta, 0), \qquad \mathbf{A}_3 = \mathbf{A}_3(\zeta) = \partial\mathbf{R}/\partial\zeta = \mathbf{a}_3(\zeta, 0),$$
$$\mathbf{D}_\alpha = \mathbf{D}_\alpha(\zeta) = \mathbf{d}_\alpha(\zeta, 0). \tag{2.4}$$

We assume that the kinetic energy of the rod per unit length of c is given by

$$T = \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v} + 2y^{0\beta}\mathbf{v}\cdot\mathbf{w}_\beta + y^{\alpha\beta}\mathbf{w}_\alpha\cdot\mathbf{w}_\beta), \tag{2.5}$$

where $\rho = \rho(\zeta, t)$ is the mass per unit length of c and the inertia coefficients $y^{0\beta} = y^{\beta 0}$, $y^{\alpha\beta} = y^{\beta\alpha}$ are functions of ζ and independent of t. We define momenta per unit length of c, corresponding to $\mathbf{v}$ and $\mathbf{w}_\alpha$, as

$$\frac{\partial T}{\partial \mathbf{v}} = \rho(\mathbf{v} + y^{0\beta}\mathbf{w}_\beta), \qquad \frac{\partial T}{\partial \mathbf{w}_\alpha} = \rho(y^{0\alpha}\mathbf{v} + y^{\alpha\beta}\mathbf{w}_\beta), \tag{2.6}$$

respecitvely. In (2.5), (2.6) and throughout the paper, we use the summation convention for repeated Greek indices over the values 1, 2.

With reference to the present configuration at time t, for each part of c between $\zeta = \alpha$, $\zeta = \beta$ we postulate the equations of mass conservation, momentum, director momentum and moment of momentum as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho\,\mathrm{d}s = 0, \tag{2.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho(\mathbf{v} + y^{0\beta}\mathbf{w}_\beta)\,\mathrm{d}s = \int_\alpha^\beta \rho\mathbf{f}\,\mathrm{d}s + [\mathbf{n}]_\alpha{}^\beta, \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho(y^{0\alpha}\mathbf{v} + y^{\alpha\beta}\mathbf{w}_\beta)\,\mathrm{d}s = \int_\alpha^\beta (\rho\mathbf{l}^\alpha - a_{33}^{-1/2}\boldsymbol{\pi}^\alpha)\,\mathrm{d}s + [\mathbf{p}^\alpha]_\alpha{}^\beta, \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho\{\mathbf{r}\times(\mathbf{v} + y^{0\beta}\mathbf{w}_\beta) + \mathbf{d}_\alpha\times(y^{0\alpha}\mathbf{v} + y^{\alpha\beta}\mathbf{w}_\beta)\}\,\mathrm{d}s = \int_\alpha^\beta \rho(\mathbf{r}\times\mathbf{f} + \mathbf{d}_\alpha\times\mathbf{l}^\alpha)\,\mathrm{d}s + [\mathbf{r}\times\mathbf{n} + \mathbf{d}_\alpha\times\mathbf{p}^\alpha]_\alpha{}^\beta, \tag{2.10}$$

where in the above integrals the limits are for values of ζ equal to α and β,

$$\mathrm{d}s = a_{33}^{1/2}\,\mathrm{d}\zeta, \qquad a_{33} = \mathbf{a}_3\cdot\mathbf{a}_3 \tag{2.11}$$

and where we have used the notation

$$[f(\zeta, t)]_\alpha{}^\beta = f(\beta, t) - f(\alpha, t).$$

Also, in (2.8) to (2.10) $\mathbf{n} = \mathbf{n}(\zeta, t)$ is the contact force and $\mathbf{p}^\alpha = \mathbf{p}^\alpha(\zeta, t)$ are the contact director forces,† each a three-dimensional field in the present configuration; $\mathbf{f} = \mathbf{f}(\zeta, t)$ is the assigned force and $\mathbf{l}^\alpha = \mathbf{l}^\alpha(\zeta, t)$ are the assigned director forces, each a three-dimensional vector field and per unit mass of c; $\boldsymbol{\pi}^\alpha = \boldsymbol{\pi}^\alpha(\zeta, t)$ are the intrinsic director forces which make no contribution to the supply of momentum and to the moment of momentum.

Under suitable smoothness assumptions the field equations corresponding to (2.7)–(2.10) are

$$\rho a_{33}^{1/2} = \lambda(\zeta) = \rho_0 A_{33}^{1/2}, \tag{2.12}$$

$$\partial\mathbf{n}/\partial\zeta + \lambda\mathbf{f} = \lambda(\dot{\mathbf{v}} + y^{0\beta}\dot{\mathbf{w}}_\beta), \tag{2.13}$$

†If $\mathbf{d}_\alpha$ have the same dimensions as $\mathbf{r}$, then $\mathbf{n}$ and $\mathbf{p}^\alpha$ have the same dimensions. On the other hand, if the directors are chosen to be dimensionless, then $\mathbf{p}^\alpha$ are usually called contact director couples.

$$\partial \mathbf{p}^\alpha/\partial\zeta + \lambda \mathbf{l}^\alpha = \boldsymbol{\pi}^\alpha + \lambda(y^{0\alpha}\dot{\mathbf{v}} + y^{\alpha\beta}\dot{\mathbf{w}}_\beta), \tag{2.14}$$

$$\mathbf{a}_3 \times \mathbf{n} + \lambda \mathbf{d}_\alpha \times (\mathbf{l}^\alpha - y^{0\alpha}\dot{\mathbf{v}} - y^{\alpha\beta}\dot{\mathbf{w}}_\beta) + \partial \mathbf{m}/\partial\zeta = \mathbf{0}, \tag{2.15}$$

where

$$\mathbf{m} = \mathbf{d}_\alpha \times \boldsymbol{p}^\alpha, \tag{2.16}$$

$\rho_0 = \rho_0(\zeta)$ is the reference density and $A_{33} = \mathbf{A}_3 \cdot \mathbf{A}_3$ is the dual of $(2.11)_2$.

3. THERMAL PROPERTIES. THERMODYNAMICAL THEORY OF RODS

In existing works on the theory of a Cosserat (or directed) curve, only one temperature field is admitted and this is regarded as representing the temperature variation along some reference curve, such as the line of centroids of the cross section of the rod-like body. Also, the effect of the thermal boundary conditions on the major surface of the rod-like body† is incorporated into the theory through the external curve rate of supply of heat. The variations of the temperature across the rod cross section have not been modelled so far by a direct approach (within the scope of the theory of directed curves), although some indications of how this could be effected are implicit in some work on thermoelastic rods from the three-dimensional equations by the present authors[6]. As already noted in Section 1, because of the new approach to thermomechanics of continua introduced recently by Green and Naghdi[1], it is now possible to account in a more general manner for the thermal properties of a rod-like body in a direct formulation based on a Cosserat curve.

Thus, at each material point of the material curve of $\mathcal{R}$, we introduce scalar fields $\theta = \theta(\zeta, t)$ and $\theta_{\alpha_1\alpha_2\ldots\alpha_N} = \theta_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ $(N = 1, \ldots, K)$ representing the effects of temperature variation in a rod-like body. The curve temperature θ, which we require to be positive $(\theta > 0)$, represents the absolute temperature in the curve c of the rod-like body, while the scalars $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ account for temperature variations across the cross section of the rod; the scalars $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ are assumed to be completely symmetric in the indices $\alpha_1, \alpha_2, \ldots, \alpha_N$ which take the values 1, 2 only. Along with the temperatures θ and $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ we admit the existence‡ of external rates of supply of heat $r = r(\zeta, t)$, $r_{\alpha_1\alpha_2\ldots\alpha_N} = r_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ per unit mass of c and external rates of heat fluxes $\bar{h}$, $\bar{h}_{\alpha_1\alpha_2\ldots\alpha_N}$ and $\bar{\bar{h}}$, $\bar{\bar{h}}_{\alpha_1\alpha_2\ldots\alpha_N}$ over each end section of the rod. Also, we assume the existence of internal heat fluxes $h = h(\zeta, t)$, $h_{\alpha_1\alpha_2\ldots\alpha_N} = h_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ along the rod at each point ζ, in the direction of increasing ζ, per unit length per unit time. Each function $r_{\alpha_1\alpha_2\ldots\alpha_N}, \ldots, h_{\alpha_1\alpha_2\ldots\alpha_N}$ is completely symmetric in the indices $\alpha_1, \alpha_2, \ldots, \alpha_N$. The total external rate of supply of heat per unit mass of c is defined as

$$r + \sum_{\alpha, N} r_{\alpha_1\alpha_2\ldots\alpha_N}, \tag{3.1}$$

where the summation in (3.1) is over all values of $\alpha_1, \alpha_2, \ldots, \alpha_N = 1, 2$ and for all $N = 1, 2, \ldots, K$. Similarly, the total internal heat flux at the point ζ is defined by

$$h + \sum_{\alpha, N} h_{\alpha_1\alpha_2\ldots\alpha_N}. \tag{3.2}$$

We now define the ratios of the heat supplies r and $r_{\alpha_1\alpha_2\ldots\alpha_N}$ to temperatures θ and $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$, respectively, as $s = s(\zeta, t)$ and $s_{\alpha_1\alpha_2\ldots\alpha_N} = s_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ and call these the *external rates of supply of entropy* per unit mass of c. Further, we define the ratios of $\bar{h}$, $\bar{\bar{h}}$ to θ and $\bar{h}_{\alpha_1\alpha_2\ldots\alpha_N}$, $\bar{\bar{h}}_{\alpha_1\alpha_2\ldots\alpha_N}$ to $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$, respectively, as the *external entropy fluxes* $\bar{k}$, $\bar{\bar{k}}$, $\bar{k}_{\alpha_1\alpha_2\ldots\alpha_N}$, $\bar{\bar{k}}_{\alpha_1\alpha_2\ldots\alpha_N}$ over the ends of the curve c. Similarly, we define the ratios of h and $h_{\alpha_1\alpha_2\ldots\alpha_N}$ to the temperatures θ and $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$, respectively, as the *internal entropy fluxes* $k = k(\zeta, t)$ and $k_{\alpha_1\alpha_2\ldots\alpha_N} = k_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ per unit length of c, in the direction of increasing ζ. The above definitions may conveniently be summarized by

†The terminology of major surface refers to the surface specified by (6.4) in Section 6.

‡The external rates of supply of heat r and $r_{\alpha_1\alpha_2\ldots\alpha_N}$ include contributions corresponding to heat fluxes on the major surfaces of the rod. They are not the same as quantities defined with a similar notation in Green and Naghdi[6].

$$
\begin{aligned}
s &= r/\theta, & s_{\alpha_1\alpha_2\ldots\alpha_N} &= r_{\alpha_1\alpha_2\ldots\alpha_N}/\theta_{\alpha_1\alpha_2\ldots\alpha_N},\\
\bar{k} &= \bar{h}/\theta, & \bar{k}_{\alpha_1\alpha_2\ldots\alpha_N} &= \bar{h}_{\alpha_1\alpha_2\ldots\alpha_N}/\theta_{\alpha_1\alpha_2\ldots\alpha_N},\\
\bar{\bar{k}} &= \bar{\bar{h}}/\theta, & \bar{\bar{h}}_{\alpha_1\alpha_2\ldots\alpha_N} &= \bar{\bar{h}}_{\alpha_1\alpha_2\ldots\alpha_N}/\theta_{\alpha_1\alpha_2\ldots\alpha_N},\\
k &= h/\theta, & k_{\alpha_1\alpha_2\ldots\alpha_N} &= h_{\alpha_1\alpha_2\ldots\alpha_N}/\theta_{\alpha_1\alpha_2\ldots\alpha_N}.
\end{aligned}
\tag{3.3}
$$

We require that the fields $s_{\alpha_1\alpha_2\ldots\alpha_N}$, $\bar{k}_{\alpha_1\alpha_2\ldots\alpha_N}$, $\bar{\bar{k}}_{\alpha_1\alpha_2\ldots\alpha_N}$, $k_{\alpha_1\alpha_2\ldots\alpha_N}$ defined by (3.3) all tend to finite limits as $\theta_{\alpha_1\alpha_2\ldots\alpha_N}\to 0$ for each $N = 1, 2, \ldots, K$.

In addition to the thermal fields already introduced, at each point of the material curve of $\mathscr{R}$ in the present configuration, we assume the existence of scalar fields $\eta = \eta(\zeta, t)$ and $\eta_{\alpha_1\alpha_2\ldots\alpha_N} = \eta_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ called *specific entropies* and *internal rates of production of entropies* $\xi = \xi(\zeta, t)$, $\xi_{\alpha_1\alpha_2\ldots\alpha_N} = \xi_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta, t)$ per unit mass of c, where $\eta_{\alpha_1\alpha_2\ldots\alpha_N}$ and $\xi_{\alpha_1\alpha_2\ldots\alpha_N}$ are completely symmetric in the indices $\alpha_1, \alpha_2, \ldots, \alpha_N$. The contributions of these internal rates of production of entropies to the rate of production of heat is

$$
\theta\xi + \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N}\xi_{\alpha_1\alpha_2\ldots\alpha_N}
\tag{3.4}
$$

per unit mass.

We postulate balances of entropies for every material curve of $\mathscr{R}$ occupying a part $\alpha \leqq \zeta \leqq \beta$ in the present configuration and write†

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho\eta \,\mathrm{d}s = \int_\alpha^\beta \rho(s+\xi)\,\mathrm{d}s - [k]_\alpha^\beta,
\tag{3.5}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho\eta_{\alpha_1\alpha_2\ldots\alpha_N} \,\mathrm{d}s = \int_\alpha^\beta \rho(s_{\alpha_1\alpha_2\ldots\alpha_N}+\xi_{\alpha_1\alpha_2\ldots\alpha_N})\,\mathrm{d}s - [k_{\alpha_1\alpha_2\ldots\alpha_N}]_\alpha^\beta.
\tag{3.6}
$$

Under suitable smoothness assumptions it follows from (3.5), (3.6), $(2.11)_1$ and (2.12) that

$$
\begin{aligned}
\lambda\dot{\eta} &= \lambda(s+\xi) - \partial k/\partial\zeta,\\
\lambda\dot{\eta}_{\alpha_1\alpha_2\ldots\alpha_N} &= \lambda(s_{\alpha_1\alpha_2\ldots\alpha_N}+\xi_{\alpha_1\alpha_2\ldots\alpha_N}) - \partial k_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta.
\end{aligned}
\tag{3.7}
$$

We now introduce the first law of thermodynamics or the balance of energy for the Cosserat curve $\mathscr{R}$. This states that heat and mechanical energy are equivalent and that together they are conserved for every part of the material curve of $\mathscr{R}$. Thus, with reference to the present configuration, the balance of energy may be stated in the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho\Big[\epsilon + \frac{1}{2}(\mathbf{v}\cdot\mathbf{v} + 2y^{0\alpha}\mathbf{v}\cdot\mathbf{w}_\alpha + y^{\alpha\beta}\mathbf{w}_\alpha\cdot\mathbf{w}_\beta)\Big]\,\mathrm{d}s &= \int_\alpha^\beta \rho\Big[r + \sum_{\alpha,N} r_{\alpha_1\alpha_2\ldots\alpha_N} + \mathbf{f}\cdot\mathbf{v} + \mathbf{l}^\alpha\cdot\mathbf{w}_\alpha\Big]\,\mathrm{d}s\\
&\quad + \Big[\mathbf{n}\cdot\mathbf{v} + \mathbf{p}^\alpha\cdot\mathbf{w}_\alpha - h - \sum_{\alpha,N} h_{\alpha_1\alpha_2\ldots\alpha_N}\Big]_\alpha^\beta,
\end{aligned}
\tag{3.8}
$$

where $\epsilon = \epsilon(\zeta, t)$ is the internal energy per unit mass of c and repeated indices are summed over the values 1, 2. With the help of (3.7), (2.12)–(2.16) and under suitable smoothness assumptions, the field equation resulting from (3.8) is

$$
\begin{aligned}
\lambda\Big(-\dot{\epsilon} + \theta\dot{\eta} + \sum_{\alpha,N}\theta_{\alpha_1\alpha_2\ldots\alpha_N}\dot{\eta}_{\alpha_1\alpha_2\ldots\alpha_N}\Big) - \lambda\Big(\theta\xi + \sum_{\alpha,N}\theta_{\alpha_1\alpha_2\ldots\alpha_N}\xi_{\alpha_1\alpha_2\ldots\alpha_N}\Big)&\\
+ P - k\partial\theta/\partial\zeta - \sum_{\alpha,N} k_{\alpha_1\alpha_2\ldots\alpha_N}\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta &= 0,
\end{aligned}
\tag{3.9}
$$

†A motivation for postulating (3.5)–(3.6) for balances of entropies is provided by consideration of derivations from three-dimensional equations in Section 7.

where the mechanical power P per unit mass of c is defined by

$$P = \mathbf{n} \cdot \partial \mathbf{v}/\partial \zeta + \boldsymbol{\pi}^\alpha \cdot \mathbf{w}_\alpha + \mathbf{p}^\alpha \cdot \partial \mathbf{w}_\alpha/\partial \zeta. \tag{3.10}$$

Introducing the Helmholtz free energy function $\psi = \psi(\zeta, t)$ per unit mass of c by

$$\psi = \epsilon - \theta\eta - \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N}\eta_{\alpha_1\alpha_2\ldots\alpha_N}, \tag{3.11}$$

the energy equation (3.9) may be written in the alternative form

$$-\lambda\Big(\dot{\psi} + \eta\dot{\theta} + \sum_{\alpha,N} \eta_{\alpha_1\alpha_2\ldots\alpha_N}\dot{\theta}_{\alpha_1\alpha_2\ldots\alpha_N}\Big) - \lambda\Big(\theta\xi + \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N}\xi_{\alpha_1\alpha_2\ldots\alpha_N}\Big)$$
$$+ P - k\partial\theta/\partial\zeta - \sum_{\alpha,N} k_{\alpha_1\alpha_2\ldots\alpha_N}\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta = 0. \tag{3.12}$$

For a given Cosserat curve having a reference density $\rho_0(\zeta)$, the field equations obtained from the integral form of the conservation laws involve a set of $(5/2)(K+1)(K+2)+12$ functions. These consist of the deformation functions r, $\mathbf{d}_\alpha$ and the temperatures θ, $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$, i.e.

$$\{\mathbf{r}, \mathbf{d}_\alpha, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}\}, \tag{3.13}$$

the various mechanical and thermal fields, namely†

$$\{\mathbf{n}, \mathbf{p}^\alpha, \boldsymbol{\pi}^\alpha, k, k_{\alpha_1\alpha_2\ldots\alpha_N}, \psi, \eta, \eta_{\alpha_1\alpha_2\ldots\alpha_N}, \xi, \xi_{\alpha_1\alpha_2\ldots\alpha_N}\} \tag{3.14}$$

and

$$\{\mathbf{f}, \mathbf{l}^\alpha, s, s_{\alpha_1\alpha_2\ldots\alpha_N}\}. \tag{3.15}$$

We assume that the fields (3.14) are specified by constitutive equations which may depend on the variables (3.13), their space and time derivatives, as well as the whole history of deformation and temperature. We then adopt the following procedure in utilizing the conservation laws:‡

(1) The field equations are assumed to hold for arbitrary choice of the functions (3.13) including, if required, an arbitrary choice of the space and time derivatives of these functions;

(2) The fields (3.14) are calculated from their respective constitutive equations;

(3) The values of the variables (3.15) can then be found from the balances of momenta (2.13) and (2.14) and balances of entropy (3.7);

(4) The eqn (2.15) resulting from the balance of moment of momentum, and the eqn (3.12) resulting from the energy equation, will be regarded as identities for every choice of (3.13). This will place restrictions on constitutive equations.

We note that the quantities ξ, $\xi_{\alpha_1\alpha_2\ldots\alpha_N}$, η, $\eta_{\alpha_1\alpha_2\ldots\alpha_N}$, ψ may be arbitrary to the extent of additive functions $\dot{f}$, $\dot{f}_{\alpha_1\alpha_2\ldots\alpha_N}$, f, $f_{\alpha_1\alpha_2\ldots\alpha_N}$, $-\theta f - \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N} f_{\alpha_1\alpha_2\ldots\alpha_N}$, respectively, where f, $f_{\alpha_1\alpha_2\ldots\alpha_N}$ are arbitrary functions of the variables (3.13), their space and time derivatives and functionals of their histories. The additive functions have the property that they make no contribution to the differential equations for $\mathbf{r}$, $\mathbf{d}_\alpha$, θ, $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ and the boundary and initial conditions. They also make no contribution to the energy identity (3.12) and no contribution to the internal energy ϵ. We remove this arbitrariness by setting

$$f = \hat{f}(\zeta), \qquad f_{\alpha_1\alpha_2\ldots\alpha_N} = \hat{f}_{\alpha_1\alpha_2\ldots\alpha_N}(\zeta),$$
$$\dot{f} = 0, \qquad \dot{f}_{\alpha_1\alpha_2\ldots\alpha_N} = 0. \tag{3.16}$$

†The density ρ is not included in (3.14) and (3.15) since, given (3.13), ρ can be calculated from (2.12).

‡For a more elaborate parallel discussion in the context of the three-dimensional theory, see Green and Naghdi [1, Section 2].

Then, the functions ξ, $\xi_{\alpha_1\alpha_2\ldots\alpha_N}$ are determined uniquely and η, $\eta_{\alpha_1\alpha_2\ldots\alpha_N}$ are only arbitrary to the extent of additive functions of ζ, independent of t. The functions $\hat{f}$, $\hat{f}_{\alpha_1\alpha_2\ldots\alpha_N}$ in (3.16) can then be determined by specifying values for η, $\eta_{\alpha_1\alpha_2\ldots\alpha_N}$ in some reference state.

So far no mention has been made of restrictions on constitutive equations which may arise from some form of second law of thermodynamics, usually interpreted in terms of an "entropy inequality". Before considering this and in order to gain some insight into the nature of our procedure described above we study in the next section the relatively simple case of an elastic rod.

For later use, we record the expressions for the external work and the external heat supplied to any part $\alpha \leqq \zeta \leqq \beta$ of the curve of c during the time interval $t_1 \leqq t \leqq t_2$. First, however, guided by the results of Section 4 we observe that in the case of an elastic rod the response functions ψ, η, $\eta_{\alpha_1\alpha_2\ldots\alpha_N}$, ϵ depend only on the vectors $\mathbf{a}_3$, $\mathbf{d}_\alpha$, $\partial\mathbf{d}_\alpha/\partial\zeta$ and the temperatures θ, $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ and are independent of their rates and the temperature gradients $\partial\theta/\partial\zeta$, $\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta$. Such an elastic rod will be regarded as nondissipative in a sense that will be made precise later; and in conjunction with an expression for the external mechanical work supplied to any part $\alpha \leqq \zeta \leqq \beta$ of the curve c, will be used as a basis for establishing in Section 5 an inequality representing the second law of thermodynamics for dissipative materials. Keeping this background in mind, we assume that the constitutive response functions for ϵ, η include also dependence on the set of variables $\dot{\mathbf{a}}_3$, $\dot{\mathbf{d}}_\alpha$, $\partial\dot{\mathbf{d}}_\alpha/\partial\zeta$, $\dot\theta$, $\dot\theta_{\alpha_1\alpha_2\ldots\alpha_N}$, $\partial\theta/\partial\zeta$, $\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta$ and their higher space and time derivatives and refer to this set collectively as υ. Further, let ϵ', η' denote the values of ϵ, η, respectively, when the set υ is put equal to zero in the response functions. Thus, for example,

$$
\begin{aligned}
\epsilon &= \epsilon(\mathbf{a}_3, \mathbf{d}_\alpha, \partial\mathbf{d}_\alpha/\partial\zeta, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}, \upsilon),\\
\epsilon' &= \epsilon'(\mathbf{a}_3, \mathbf{d}_\alpha, \partial\mathbf{d}_\alpha/\partial\zeta, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N})\\
&= \epsilon(\mathbf{a}_3, \mathbf{d}_\alpha, \partial\mathbf{d}_\alpha/\partial\zeta, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}, 0),\\
\upsilon &= (\dot{\mathbf{a}}_3, \dot{\mathbf{d}}_\alpha, \partial\dot{\mathbf{d}}_\alpha/\partial\zeta, \dot\theta, \dot\theta_{\alpha_1\alpha_2\ldots\alpha_N}, \partial\theta/\partial\zeta, \partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta, \ldots),
\end{aligned}
\tag{3.17}
$$

where the dots in $(3.17)_3$ refer to the higher space and time derivatives of $\dot{\mathbf{a}}_3, \ldots, \dot\theta_{\alpha_1\alpha_2\ldots\alpha_N}$. Then, with the help of (2.12)–(2.14) and the integral of (3.8) with respect to time, we obtain

$\mathcal{W}$ = External mechanical work supplied to a part $\alpha \leqq \zeta \leqq \beta$ of the rod during the time interval $t_1 \leqq t \leqq t_2$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} \left\{ \int_\alpha^\beta \rho(\mathbf{f}\cdot\mathbf{v} + \mathbf{l}^\alpha \cdot \mathbf{w}_\alpha)\, ds + [\mathbf{n}\cdot\mathbf{v} + \mathbf{p}^\alpha \cdot \mathbf{w}_\alpha]_\alpha^{\beta} \right\} dt\\
&= \Delta K + \Delta E + \bar{\mathcal{W}} + \mathcal{W}_2
\end{aligned}
\tag{3.18}
$$

and

$\mathcal{H}$ = External heat supplied to a part of $\alpha \leqq \zeta \leqq \beta$ of the rod during the time interval $t_1 \leqq t \leqq t_2$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} \left\{ \int_\alpha^\beta \rho\left(r + \sum_{\alpha,N} r_{\alpha_1\alpha_2\ldots\alpha_N}\right) ds - \left[h + \sum_{\alpha,N} h_{\alpha_1\alpha_2\ldots\alpha_N}\right]_\alpha^\beta \right\} dt\\
&= -\bar{\mathcal{W}} - \mathcal{W}_2,
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
\bar{\mathcal{W}} &= -\int_{t_1}^{t_2} \int_\alpha^\beta \rho\left(\theta\dot\eta' + \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N} \dot\eta'_{\alpha_1\alpha_2\ldots\alpha_N}\right) ds\, dt\\
\mathcal{W}_2 &= \int_{t_1}^{t_2} \int_\alpha^\beta \rho w\, ds\, dt, \qquad E = \int_\alpha^\beta \rho\epsilon\, ds,\\
K &= \int_\alpha^\beta \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v} + 2y^{0\alpha}\mathbf{v}\cdot\mathbf{w}_\alpha + y^{\alpha\beta}\mathbf{w}_\alpha\cdot\mathbf{w}_\beta)\, ds.
\end{aligned}
\tag{3.20}
$$

The prefix Δ in (3.18) denotes the difference operations on functions and fields during the time interval $[t_1, t_2]$, e.g. $\Delta E = E(t_2) - E(t_1)$. Also, w in $(3.20)_2$ is given by

$$\lambda w = P - \lambda(\dot{\epsilon} - \dot{\epsilon}') - \lambda\left(\dot{\psi}' + \eta'\dot{\theta} + \sum_{\alpha,N} \eta'_{\alpha_1\alpha_2\ldots\alpha_N}\, \dot{\theta}_{\alpha_1\alpha_2\ldots\alpha_N}\right)$$

$$= -\lambda\left[(\dot{\eta} - \dot{\eta}')\theta + \sum_{\alpha,N} (\dot{\eta}_{\alpha_1\alpha_2\ldots\alpha_N} - \dot{\eta}'_{\alpha_1\alpha_2\ldots\alpha_N})\theta_{\alpha_1\alpha_2\ldots\alpha_N}\right]$$
$$+ \lambda\left(\theta\xi + \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N}\xi_{\alpha_1\alpha_2\ldots\alpha_N}\right) + k\partial\theta/\partial\zeta + \sum_{\alpha,N} k_{\alpha_1\alpha_2\ldots\alpha_N}\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta, \tag{3.21}$$

$$\psi' = \epsilon' - \theta\eta' - \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N}\eta'_{\alpha_1\alpha_2\ldots\alpha_N}. \tag{3.22}$$

4. THERMOELASTIC RODS

A thermoelastic theory of rods by a direct approach was given by Green and Laws[3] and was developed further by Green *et al.*[7] and by Green *et al.*[5]. The previous work made use of a one-dimensional Clausius–Duhem inequality and only one temperature field was considered, which corresponds to the curve temperature θ of the present paper. We consider now constitutive equations for a thermoelastic rod which admits $(1/2)(K+1)(K+2)$ temperature fields and we examine the restrictions imposed on these equations by the procedure described at the end of Section 3.

We assume that the set of variables (3.14) are functions of

$$\mathbf{a}_3,\ \mathbf{d}_\alpha,\ \partial\mathbf{d}_\alpha/\partial\zeta,\ \theta,\ \theta_{\alpha_1\alpha_2\ldots\alpha_N},\ \partial\theta/\partial\zeta,\ \partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta, \tag{4.1}$$

as well as the reference values

$$\mathbf{A}_3,\ \mathbf{D}_\alpha,\ \partial\mathbf{D}_\alpha/\partial\zeta,\ \Theta \tag{4.2}$$

and in addition may depend also on the particle ζ. In the set of reference values (4.2), Θ is the constant reference value of θ and we have assumed that the reference values of $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ $(\alpha = 1, 2;\ N = 1, 2, \ldots, K)$ are zero. Postponing the restrictions to be imposed by the invariance requirements under superposed rigid body motions and recalling the procedure outlined in Section 3, the energy eqn (3.12) is identically satisfied for all thermo-mechanical processes provided

$$\frac{\partial\psi}{\partial(\partial\theta/\partial\zeta)} = 0, \qquad \frac{\partial\psi}{\partial(\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta)} = 0, \tag{4.3}$$

$$\psi = \bar{\psi}(\mathbf{a}_3, \mathbf{d}_\alpha, \partial\mathbf{d}_\alpha/\partial\zeta, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}; \mathbf{A}_3, \mathbf{D}_\alpha, \partial\mathbf{D}_\alpha/\partial\zeta, \Theta; \zeta), \tag{4.4}$$

$$\mathbf{n} = \lambda\frac{\partial\bar{\psi}}{\partial\mathbf{a}_3}, \qquad \boldsymbol{\pi}^\alpha = \lambda\frac{\partial\bar{\psi}}{\partial\mathbf{d}_\alpha}, \qquad \mathbf{p}^\alpha = \lambda\frac{\partial\bar{\psi}}{\partial\mathbf{d}_\alpha/\partial\zeta}, \tag{4.5}$$

$$\eta = -\frac{\partial\bar{\psi}}{\partial\theta}, \qquad \eta_{\alpha_1\alpha_2\ldots\alpha_N} = -\frac{\partial\bar{\psi}}{\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}}, \tag{4.6}$$

$$\lambda\left(\theta\xi + \sum_{\alpha,N}\theta_{\alpha_1\alpha_2\ldots\alpha_N}\xi_{\alpha_1\alpha_2\ldots\alpha_N}\right) + k\partial\theta/\partial\zeta + \sum_{\alpha,N} k_{\alpha_1\alpha_2\ldots\alpha_N}(\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta) = 0, \tag{4.7}$$

where as indicated in (4.3), the function $\bar{\psi}$ is independent of temperature gradients. It is understood that $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ appears in $\bar{\psi}$ only in a form which is completely symmetric in the indices $\alpha_1\alpha_2\ldots\alpha_N$. Formulae (4.5) and (4.6), with $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ absent, were obtained by Green *et al.*[5]† with the help of the Clausius–Duhem inequality. Alternative forms for the results (4.4)–$(4.6)_1$ have been given by Green and Laws[3] and by Green *et al.*[5] after making use of invariance conditions under superposed rigid body motions. For later use we record one to

†In this paper p. 490, equation $(7.5)_4$ should read $\mathbf{p}^\alpha = \lambda\partial\bar{\psi}/\partial\mathbf{d}'_\alpha$.

these alternative forms here and for this purpose we introduce the notations

$$\begin{aligned} \mathbf{d}_3 = \mathbf{a}_3 \quad h_{ij} = \mathbf{d}_i \cdot \mathbf{d}_j, \quad \lambda_{\alpha i} = \mathbf{d}_i \cdot \partial \mathbf{d}_\alpha / \partial \zeta, \\ \mathbf{d}^i \cdot \mathbf{d}_j = \delta^i_j, \quad h^{ij} = \mathbf{d}^i \cdot \mathbf{d}^j, \quad h = \det h_{ij}, \end{aligned} \tag{4.8}$$

where Latin indices take the values 1, 2, 3 and, from (2.1), we recall the condition

$$h^{1/2} = [\mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3] > 0, \tag{4.9}$$

so that $h^{1/2}$ is a single-valued function of h_{ij}. For the duals of quantities in (4.8) defined relative to a reference configuration, we employ the corresponding capital letters, e.g. $\Lambda_{i\alpha}$, H_{ij}, etc.

Under superposed rigid body motions the vectors $\mathbf{a}_3, \mathbf{d}_\alpha, \partial \mathbf{d}_\alpha/\partial\zeta$ become $\mathbf{Q}\mathbf{a}_3$, $\mathbf{Q}\mathbf{d}_\alpha$, $\mathbf{Q}(\partial \mathbf{d}_\alpha/\partial\zeta)$, where $\mathbf{Q}$ is a proper orthogonal tensor function of the time, and the corresponding value ψ^+ of the free energy response is given by

$$\psi^+ = \bar{\psi}(Q\mathbf{a}_3, Q\mathbf{d}_\alpha, \mathbf{Q}\partial \mathbf{d}_\alpha/\partial\zeta, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}; \mathbf{A}_3, \mathbf{D}_\alpha, \partial \mathbf{D}_\alpha/\partial\zeta, \Theta; \zeta). \tag{4.10}$$

Then, by (4.4), (4.10) and the invariance condition $\psi^+ = \psi$, we have

$$\bar{\psi}(\mathbf{Q}\mathbf{a}_3, \mathbf{Q}\mathbf{d}_\alpha, \mathbf{Q}\partial \mathbf{d}_\alpha/\partial\zeta, \ldots) = \bar{\psi}(\mathbf{a}_3, \mathbf{d}_\alpha, \partial \mathbf{d}_\alpha/\partial\zeta, \ldots) \tag{4.11}$$

for all proper orthogonal tensors $\mathbf{Q}$. It follows from Cauchy's representation theorem that $\bar{\psi}$ may be expressed as a different function of the inner products and scalar triple products of $\mathbf{a}_3$, $\mathbf{d}_\alpha$, $\partial \mathbf{d}_\alpha/\partial\zeta$, namely the inner products $(4.8)_{2,3}$ and

$$[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3], \quad [\mathbf{d}_1\mathbf{d}_2\partial \mathbf{d}_\alpha/\partial\zeta], \quad [\mathbf{d}_1\mathbf{d}_3\partial \mathbf{d}_\alpha/\partial\zeta], \quad [\mathbf{d}_2\mathbf{d}_3\partial \mathbf{d}_\alpha/\partial\zeta], \quad [\mathbf{d}_i/\partial \mathbf{d}_1/\partial\zeta\partial \mathbf{d}_2/\partial\zeta]. \tag{4.12}$$

In view of (4.9), each of the scalar triple products in (4.12) may be expressed as a single-valued function of $h^{1/2}$ and $(4.8)_{2,3}$ and hence of h_{ij} and $\lambda_{\alpha i}$. Similarly, if instead of $(2.1)_3$ or (4.9) we made the choice $[\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3] < 0$ for all motions, then we may again reduce ψ to depend only on h_{ij}, $\lambda_{\alpha i}$, apart from the reference variables and temperatures. Hence, we may replace (4.4), by the different single-valued function

$$\psi = \hat{\psi}(\gamma_{ij}, \kappa_{\alpha i}, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}; \mathbf{A}_3, \mathbf{D}_\alpha, \partial \mathbf{D}_\alpha/\partial\zeta, \Theta; \zeta), \tag{4.13}$$

where

$$\gamma_{ij} = h_{ij} - H_{ij}, \quad \kappa_{\alpha i} = \lambda_{\alpha i} - \Lambda_{\alpha i}. \tag{4.14}$$

Let $\mathbf{a}_i$ be a set of base vectors with $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] \neq 0$ and let $\mathbf{a}^i$ be their reciprocals. The base vectors $\mathbf{a}_i$ ordinarily will be taken to be orthogonal but this is not essential at this point in our development. It is now convenient to introduce the component forms of the kinematic variables $\mathbf{d}_i, \mathbf{d}^i$ and of the kinetic variables relative to the base vectors $\mathbf{a}_i$ or $\mathbf{a}^i$. Thus, we write

$$\begin{aligned} \mathbf{d}_i = d_{ij}\mathbf{a}^j = d_{i\cdot}^{\ j}\mathbf{a}_j, \quad \mathbf{d}^i = d^{ij}\mathbf{a}_j = d^{i\cdot}_{\ j}\mathbf{a}^j, \\ \mathbf{n} = n^i\mathbf{a}_i, \quad \boldsymbol{\pi}^\alpha = \pi^{\alpha i}\mathbf{a}_i, \quad \mathbf{p}^\alpha = p^{\alpha i}\mathbf{a}_i. \end{aligned} \tag{4.15}$$

It then follows from (4.5), (4.6) and (4.13) that

$$\begin{aligned} d^{3\cdot}_{\ i}(n^i - \lambda_{\nu\cdot}^{\ 3}p^{\nu i}) = 2\lambda \frac{\partial\hat{\psi}}{\partial\gamma_{33}}, \quad d^{\alpha\cdot}_{\ i}n^i - d^{3\cdot}_{\ i}\lambda_{\nu\cdot}^{\ \alpha}p^{\nu i} = \lambda\frac{\partial\hat{\psi}}{\partial\gamma_{\alpha 3}}, \\ d^{\beta\cdot}_{\ i}(\pi^{\alpha i} - \lambda_{\nu\cdot}^{\ \alpha}p^{\nu i}) + d^{\alpha\cdot}_{\ i}(\pi^{\beta i} - \lambda_{\nu\cdot}^{\ \beta}p^{\nu i}) = 4\lambda\frac{\partial\hat{\psi}}{\partial\gamma_{\alpha\beta}}, \\ p^{\alpha i} = \lambda d_{j\cdot}^{\ i}\frac{\partial\hat{\psi}}{\partial\kappa_{\alpha j}}, \\ \eta = -\frac{\partial\hat{\psi}}{\partial\theta}, \quad \eta_{\alpha_1\alpha_2\ldots\alpha_N} = -\frac{\partial\psi}{\partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}} \end{aligned} \tag{4.16}$$

and in evaluating $(4.16)_{1,2,3}$, $\hat{\psi}$ is regarded as a function of γ_{33}, $\gamma_{\alpha 3}$ and $(1/2)(\gamma_{\alpha\beta}+\gamma_{\beta\alpha})$. The results (4.16) are equivalent to (7.40) in Green *et al.*[5]; those in Green and Laws[3] are a special case of (4.16) when $\mathbf{d}_i = \mathbf{a}_i$, apart from some changes in notation and definitions.

With the help of (2.14) and (2.16), the moment of momentum equation (2.15) can be expressed in the form

$$\mathbf{a}_3 \times \mathbf{n} + \mathbf{d}_\alpha \times \boldsymbol{\pi}^\alpha + (\partial \mathbf{d}_\alpha / \partial \zeta) \times \mathbf{p}^\alpha = \mathbf{0}. \tag{4.17}$$

This equation is identically satisfied by the expressions (4.16). This implies that the only relevant field equations are (2.13) and (2.14).

In a similar manner, with the help of invariance conditions under superposed rigid body motions, the entropy flux functions $k, k_{\alpha_1\alpha_2\ldots\alpha_N}$ may be reduced to depend on the variables displayed in (4.13).

With the help of (4.5) and (4.6) we see that the expression for w in (3.21) is zero and the external mechanical work $\mathscr{W}$ and external heat supplied $\mathscr{H}$ in (3.18) and (3.19) reduce to

$$\mathscr{W} = \Delta K + \Delta E' + \bar{\mathscr{W}} \tag{4.18}$$

and

$$\mathscr{H} = -\bar{\mathscr{W}}, \tag{4.19}$$

where

$$E' = \int_\alpha^\beta \rho \epsilon' \, \mathrm{d}s \tag{4.20}$$

and K and $\bar{\mathscr{W}}$ are defined in (3.20).

5. THE SECOND LAW OF THERMODYNAMICS

Previously, in the context of the three-dimensional theory, Green and Naghdi[1, 2] have discussed the nature of thermodynamic irreversibility arising from a mathematical interpretation of a statement of the second law namely that "it is impossible completely to reverse a process in which energy is transformed into heat by friction". Here we follow the same procedure and reconsider a mathematical interpretation of a second law appropriate for a direct theory of rods which admits more than one temperature field. In earlier works[3, 5], a Clausius–Duhem inequality was used and only one temperature field was admitted.

The state of a rod at time t which is regarded as representing a thin rod-like three-dimensional body, is described by the position vector $\mathbf{r}$ and the directors $\mathbf{d}_\alpha$, the velocities $\mathbf{v}, \mathbf{w}_\alpha$, the temperatures $\theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}$ $(N = 1, 2, \ldots, K)$ throughout the curve c, together with the constitutive response functions for the fields (3.14). A thermo-mechanical process or simply a process is a time sequence of states: it is a continuous oriented curve in the space of states, i.e. the $(\theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}, h_{ij}, \lambda_{\alpha i})$ space. Thus, a process may be specified by a sequence of values of

$$\mathbf{r}, \mathbf{d}_\alpha, \theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N} \tag{5.1}$$

on c in the time interval $0 \leqq t \leqq \sigma$. Similarly, the reverse process is a process defined by a sequence of values of (5.1) on c in the time interval $\sigma \leqq t \leqq 2\sigma$ subject to the conditions

$$\begin{aligned} \mathbf{r}(t) = \mathbf{r}(2\sigma - t), \qquad \mathbf{d}_\alpha(t) = \mathbf{d}_\alpha(2\sigma - t), \\ \theta(t) = \theta(2\sigma - t), \qquad \theta_{\alpha_1\alpha_2\ldots\alpha_N}(t) = \theta_{\alpha_1\alpha_2\ldots\alpha_N}(2\sigma - t). \end{aligned} \tag{5.2}$$

In any process the work done by the external mechanical forces acting on a part $\alpha \leqq \zeta \leqq \beta$ of c, and given by (3.18), is a positive or negative depending on whether the external work is supplied to, or is withdrawn from, the part. In general, some of the work done results in a change of the kinetic and internal energies represented by the first two terms on the right-hand side of $(3.18)_3$, each of which may be positive, negative or zero. Also part of the work done may be positive with a corresponding absorption of heat by the part $\alpha \leqq \zeta \leqq \beta$ or negative with a corresponding absorption of heat by the part. We note that in the case of an elastic material the different contributions to $\mathscr{W}$ will vary in sign depending on the process and will not be restricted to be either positive or negative for all processes. Consider any process in the time interval $0 \leqq t \leqq \sigma$ and its reverse process in the time interval $\sigma \leqq t \leqq 2\sigma$. If the process is reversed in such a way that at the end of the process and its reverse process the elastic rod has returned to its original state with $\Delta\theta = 0$, $\Delta\theta_{\alpha_1\alpha_2\ldots\alpha_N} = 0$, $\Delta h_{ij} = 0$, $\Delta\lambda_{\alpha i} = 0$, $\Delta \mathbf{v} = \mathbf{0}$, $\Delta \mathbf{w}_\alpha = \mathbf{0}$ and, hence, $\Delta\epsilon = 0$, $\Delta\eta = 0$, $\Delta\eta_{\alpha_1\alpha_2\ldots\alpha_N} = 0$, $\Delta\xi = 0$, $\Delta\xi_{\alpha_1\alpha_2\ldots\alpha_N} = 0$, $\Delta\mathbf{n} = \mathbf{0}$, $\Delta\mathbf{p}^\alpha = \mathbf{0}$, $\Delta\boldsymbol{\pi}^\alpha = \mathbf{0}$, $\Delta k = 0$, $\Delta k_{\alpha_1\alpha_2\ldots\alpha_N} = 0$ and $\Delta K = 0$, $\Delta E = 0$, then all the work done in the process is recovered as work in the reverse process.† This recovery of work would not be possible if in every arbitrary process part of $\mathscr{W}$ always has a positive sign. With this motivation in mind, we assume that for any arbitrary process in a dissipative rod only part of the work done is recoverable as work in the reverse process, the rest being transformed into heat. We therefore assume that in every process part of the work done is always nonnegative. Then, if at the end of any process and its reverse process the rod has returned to the same state, some of the work done is always transformed into heat. Recalling that $\mathscr{W}_2 = 0$ in (3.18) in the case of an elastic rod, we interpret the above assumption for a dissipative rod by requiring that

$$\mathscr{W}_2 \geqq 0 \tag{5.3}$$

for all parts of c and all processes, where $\mathscr{W}_2$ is given by (3.20). Since t_1, t_2 are arbitrary and w has already been assumed to be continuous, it follows that

$$\lambda w = P - \lambda(\dot{\epsilon} - \dot{\epsilon}') - \lambda\Big(\dot{\psi}' + \eta'\dot{\theta} + \sum_{\alpha,N} \eta'_{\alpha_1\alpha_2\ldots\alpha_N}\dot{\theta}_{\alpha_1\alpha_2\ldots\alpha_N}\Big) \geqq 0 \tag{5.4}$$

for all thermo-mechanical processes. Also, from (3.19) and (5.4), we have

$$\mathscr{H} \leqq \int_{t_1}^{t_2}\int_\alpha^\beta \lambda\Big(\theta\dot{\eta}' + \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N}\dot{\eta}'_{\alpha_1\alpha_2\ldots\alpha_N}\Big)\,\mathrm{d}s\,\mathrm{d}t \tag{5.5}$$

so that the external heat supplied to any part of c is bounded above in any process.

6. SUMMARY OF RESULTS FROM THREE-DIMENSIONAL MECHANICAL THEORY

Consider a three-dimensional body, embedded in Euclidean 3-space, and let its particles be identified by convected coordinates ζ^i $(i = 1, 2, 3)$. Let $\mathbf{r}^*$ denote the position vector, relative to a fixed origin, of a typical particle of the three-dimensional body in the present configuration at time t. Then,

$$\begin{gathered}\mathbf{r}^* = \mathbf{r}^*(\zeta^1, \zeta^2, \zeta^3, t), \qquad \mathbf{g}_i = \partial\mathbf{r}^*/\partial\zeta^i, \qquad \mathbf{v}^* = \dot{\mathbf{r}}^*, \\ g_{ik} = \mathbf{g}_i \cdot \mathbf{g}_k, \qquad \mathbf{g}^i \cdot \mathbf{g}_k = \delta_k^{\,i}, \qquad g^{ik} = \mathbf{g}^i \cdot \mathbf{g}^k, \qquad g = \det g_{ik},\end{gathered} \tag{6.1}$$

where $\mathbf{g}_i$ and $\mathbf{g}^i$ are covariant and contravariant base vectors, respectively, g_{ik} and g^{ik} are covariant and contravariant metric tensors, respectively, and $\delta_k^{\,i}$ is the Kronecker delta. Also a superposed dot dentoes material time derivative holding ζ^i fixed and $\mathbf{v}^*$ is the velocity vector.

The stress vector $\mathbf{t}$ across a surface in the present configuration whose unit outward normal is $\boldsymbol{\nu}^*$ may be expressed in the form

$$\mathbf{t} = \nu_i^*\mathbf{T}^i/g^{1/2} = \nu_i^*\tau^{ik}\mathbf{g}_k, \qquad \boldsymbol{\nu}^* = \nu_i^*\mathbf{g}^i = \nu^{*i}\mathbf{g}_i, \tag{6.2}$$

where τ^{ik} are the contravariant components of the symmetric stress tensor. We do not recall here the consequences of the conservation laws of the three-dimensional theory since they will not be needed in the preent paper.

The parametric equations $\zeta^\alpha = 0$ define a curve in space at time t which we assume to be

†If work is extracted in the process then it is absorbed by the rod in the reverse process.

smooth and which we identify with the curve c. Any point of c is specified by the position vector $\mathbf{r}$ relative to the same fixed origin to which $\mathbf{r}^*$ is referred, where

$$\mathbf{r} = \mathbf{r}(\zeta, t) = \mathbf{r}^*(0, 0, \zeta, t), \qquad \zeta^3 = \zeta. \tag{6.3}$$

Let the boundary of the three-dimensional region occupied by the body at time t be specified by the material surface

$$F(\zeta^1, \zeta^2, \zeta) = 0 \tag{6.4}$$

and by the surfaces $\zeta = \alpha, \beta$. The surface (6.4) is such that ζ = constant are curved sections of the body bounded by closed curves on this surface.

Suppose now that $\mathbf{r}^*$ in $(6.1)_1$ is a continuous function of ζ^i, t and has continuous space and time derivatives of order 2 in the bounded region lying inside the surface (6.4) and between $\zeta = \alpha, \beta$. Hence, to any required degree of approximation, $\mathbf{r}^*$ may be represented as a polynomial in ζ^1, ζ^2 with coefficients which are twice continuously differentiable functions of ζ, t. Instead of considering a general representation of this kind, we restrict attention here to the approximation

$$\mathbf{r}^* = \mathbf{r}(\zeta, t) + \zeta^\alpha \mathbf{d}_\alpha(\zeta, t). \tag{6.5}$$

Given the approximation (6.5) it is known (see, e.g. Green and Naghdi [6]; Green *et al.* [7]) that the field equations of the forms (2.12) to (2.15) can be derived from the three-dimensional equations provided we identify $\mathbf{d}_\alpha$ in (6.5) with $(2.1)_2$ and adopt the definitions

$$\begin{gathered} \rho a_{33}^{1/2} = \lambda = \iint \mu \, d\zeta^1 \, d\zeta^2, \qquad \mu = \rho^* g^{1/2} = \mu(\zeta^1, \zeta^2), \\ \lambda y^{\alpha\beta} = \iint \mu \zeta^\alpha \zeta^\beta \, d\zeta^1 \, d\zeta^2 \end{gathered} \tag{6.6}$$

and

$$\begin{gathered} \mathbf{n} = \iint \mathbf{T}^3 \, d\zeta^1 \, d\zeta^2, \qquad \boldsymbol{\pi}^\alpha = \iint \mathbf{T}^\alpha \, d\zeta^1 \, d\zeta^2, \\ \mathbf{p}^\alpha = \iint \zeta^\alpha \mathbf{T}^3 \, d\zeta^2 \, d\zeta^2, \end{gathered} \tag{6.7}$$

where ρ^* is the three-dimensional mass density and the integrals are taken over any surface ζ = constant bounded by (6.4). For some purposes it is convenient to define the curve $\zeta^1 = 0$, $\zeta^2 = 0$ in relation to the surface (6.4) so that $y^{0\alpha} = 0$, but this is not essential. The assigned forces $\mathbf{f}, \mathbf{l}^\alpha$ are related to the three-dimensional body forces $\mathbf{f}^*$ and to the effects of the stress vector $(6.2)_1$ over the boundary surface (6.4) by

$$\begin{gathered} \lambda \mathbf{f} = \iint \mu \mathbf{f}^* \, d\zeta^1 \, d\zeta^2 + \oint [(\mathbf{T}^1 - \lambda^1 \mathbf{T}^3) \, d\zeta^2 - (\mathbf{T}^2 - \lambda^2 \mathbf{T}^3) \, d\zeta^1], \\ \lambda \mathbf{l}^\alpha = \iint \mu \mathbf{f}^* \zeta^\alpha \, d\zeta^1 \, d\zeta^2 + \oint \zeta^\alpha [(\mathbf{T}^1 - \lambda^1 \mathbf{T}^3) \, d\zeta^2 - (\mathbf{T}^2 - \lambda^2 \mathbf{T}^3) \, d\zeta^1], \end{gathered} \tag{6.8}$$

where the line integrals are taken along the curve

$$\zeta = \text{const.}, \qquad F(\zeta^1, \zeta^2, \zeta) = 0, \tag{6.9}$$

$\lambda^\alpha = \boldsymbol{\lambda} \cdot \mathbf{g}^\alpha$ and

$$\boldsymbol{\lambda} = \lambda^\alpha \mathbf{g}_\alpha + \mathbf{g}_3 \tag{6.10}$$

is a vector tangential to the survace (6.4) so that

$$\boldsymbol{\lambda} \cdot \boldsymbol{\nu}^* = \lambda^\alpha \nu_\alpha^* + \nu_3^* = 0. \tag{6.11}$$

7. THERMODYNAMICAL RESULTS FROM THREE-DIMENSIONAL THEORY

In this section we obtain some thermodynamical results for a rod-like body on the basis of the recent thermodynamical theory of Green and Naghdi[1]. Thus, along with the three-dimensional temperature field $\theta^* = \theta^*(\zeta^i, t) > 0$ we admit the existence of an external rate of supply of heat $-\bar{h}^*$ per unit area acting across the boundary $\partial\mathcal{R}^*$ of a region of space $\mathcal{R}^*$ occupied by the body in the present configuration at time t. Also we assume the existence of an internal surface flux of heat $-h^* = -h^*(\zeta^i, t; \boldsymbol{\nu}^*)$ per unit area across each surface $\partial\mathcal{P}^*$ which is the boundary of an arbitrary part $\mathcal{P}^*$ of $\mathcal{R}^*$. We define the ratio of the heat supply r^* to temperature θ^* as $s^* = s^*(\zeta^1, t)$ and call this the external rate of supply of entropy per unit mass. Similarly, we define the ratios of $\bar{h}^*$ and h^* to temperature, respectively, as the external rate of surface supply of entropy $\bar{k}^*$ per unit area of $\partial\mathcal{R}^*$ and the internal surface flux of entropy $k^* = k^*(\zeta^i, t; \boldsymbol{\nu}^*)$ per unit area of $\partial\mathcal{P}^*$. Thus

$$r^* = \theta^* s^*, \qquad \bar{h}^* = \theta^* \bar{k}^*, \qquad h^* = \theta^* k^*. \tag{7.1}$$

In addition, throughout $\mathcal{R}^*$ we assume the existence of a scalar field $\eta^* = \eta^*(\zeta^i, t)$ per unit mass, called the specific entropy and an internal rate of production of entropy $\xi^* = \xi^*(\zeta^i, t)$ per unit mass. The contribution of the latter to the rate of production of heat is $\theta^*\xi^*$ per unit mass.

We recall the balance of entropy in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}^*} \rho^*\eta^*\,\mathrm{d}v = \int_{\mathcal{P}^*} \rho^*(s^* + \xi^*)\,\mathrm{d}v - \int_{\partial\mathcal{P}^*} k^*\,\mathrm{d}a \tag{7.2}$$

for every material volume occupying a part $\mathcal{P}^*$ in the present configuration. It follows that k^* is linear in $\boldsymbol{\nu}^*$, i.e.

$$k^* = \mathbf{p}^* \cdot \boldsymbol{\nu}^*, \qquad \mathbf{p}^* = p^{*i}\mathbf{g}_i, \tag{7.3}$$

where $\mathbf{p}^*$ is the entropy flux vector. Then, from (7.1) and (7.3), $h^* = \theta^*\mathbf{p}^* \cdot \boldsymbol{\nu}^*$ and we may define the heat flux vector $\mathbf{q}^*$ by

$$\mathbf{q}^* = \theta^* \mathbf{p}^*. \tag{7.4}$$

With the help of (7.3), the field equation corresponding to (7.2) is

$$\rho^*\dot{\eta}^* = \rho^*(s^* + \xi^*) - \operatorname{div}\mathbf{p}^*, \tag{7.5}$$

where

$$\operatorname{div}\mathbf{p}^* = g^{-1/2}\partial(p^{*k}g^{1/2})/\partial\zeta^k. \tag{7.6}$$

Multiply (7.5) by $\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}$ and integrate over an arbitrary part $\mathcal{P}^*$ in the present configuration. After using (7.3) and some straightforward manipulation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}^*} \rho^*\eta^*\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}\,\mathrm{d}v &= \int_{\mathcal{P}^*} \rho^*(s^* + \xi^*)\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}\,\mathrm{d}v - \int_{\partial\mathcal{P}^*} k^*\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}\,\mathrm{d}a \\ &\quad + \int_{\mathcal{P}^*} (p^{*\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N} + \cdots + p^{*\alpha_N}\zeta^{\alpha_1}\ldots\zeta^{\alpha_{N-1}})\,\mathrm{d}v.\end{aligned} \tag{7.7}$$

We now suppose that $\mathcal{P}^*$ is a region bounded by the surface (6.4) and by $\zeta = \alpha, \beta$. Then for a rod-like body, from (7.2) and (7.7) we may derive the balance equations (3.5) and (3.6) without any approximations provided we make the following identifications:

$$\lambda\eta = \int \mu\eta^*\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2, \qquad \lambda\eta_{\alpha_1\alpha_2\ldots\alpha_N} = \int \mu\eta^*\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2, \tag{7.8}$$

$$\lambda s = \iint \mu s^*\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2 - \oint k^* g^{1/2}[(\nu^{*1} - \lambda^1\nu^{*3})\,\mathrm{d}\zeta^2 - (\nu^{*2} - \lambda^2\nu^{*3})\,\mathrm{d}\zeta^1], \tag{7.9}$$

$$\lambda s_{\alpha_1\alpha_2\ldots\alpha_N} = \int \mu s^* \zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2$$
$$-\oint k^* g^{1/2}\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}[(\nu^{*1}-\lambda^1\nu^{*3})\,\mathrm{d}\zeta^2-(\nu^{*2}-\lambda^2\nu^{*3})\,\mathrm{d}\zeta^1], \tag{7.10}$$

$$\lambda\xi = \iint \mu\xi^*\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2, \tag{7.11}$$

$$\lambda\xi_{\alpha_1\alpha_2\ldots\alpha_N} = \lambda\bar{\xi}_{\alpha_1\alpha_2\ldots\alpha_N} + \iint \mu\xi^*\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2,$$
$$\lambda\bar{\xi}_{\alpha_1\alpha_2\ldots\alpha_N} = \iint g^{1/2}(p^{*\alpha_1}\zeta^{\alpha_2}\zeta^{\alpha_3}\ldots\zeta^{\alpha_N}+\cdots+p^{*\alpha_N}\zeta^{\alpha_1}\ldots\zeta^{\alpha_{N-1}})\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2, \tag{7.12}$$

$$k = \iint k^*(gg^{33})^{1/2}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2 = \iint p^{*3}g^{1/2}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2,$$
$$k_{\alpha_1\alpha_2\ldots\alpha_N} = \iint k^*(gg^{33})^{1/2}\zeta^{\alpha_1}\ldots\zeta^{\alpha_N}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2 \tag{7.13}$$
$$= \iint p^{*3}g^{1/2}\zeta^{\alpha_1}\ldots\zeta^{\alpha_N}\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2.$$

We also recall the three-dimensional equation for the conservation of energy, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\left(\frac{1}{2}\mathbf{v}^*\cdot\mathbf{v}^*+\epsilon^*\right)\rho^*\,\mathrm{d}v = \int_{\mathscr{P}^*}(s^*\theta^*+\mathbf{f}^*\cdot\mathbf{v}^*)\rho^*\,\mathrm{d}v + \int_{\partial\mathscr{P}^*}(\mathbf{t}\cdot\mathbf{v}^*-k^*\theta^*)\,\mathrm{d}a, \tag{7.14}$$

where ϵ^* is the internal energy density. Suppose in addition to the approximation (6.5) for the displacement vector, we adopt the approximation

$$\theta^* = \theta + \sum_{\alpha,N}\theta_{\alpha_1\alpha_2\ldots\alpha_N}\zeta^{\alpha_1}\zeta^{\alpha_2}\ldots\zeta^{\alpha_N}, \qquad (\theta>0) \tag{7.15}$$

for the temperature field. The summation in (7.15) is over all values of $N = 1, 2, \ldots, K$ and all Greek indices have the values 1, 2. The functions $\theta, \theta_{\alpha_1\alpha_2\ldots\alpha_N}$ depend on ζ, t and $\theta_{\alpha_1\alpha_2\ldots\alpha_N}$ is completely symmetric in its suffices. Then, for a rod-like region bounded by the surfaces (6.4) and $\zeta = \alpha, \beta$, from the energy equation (7.14) we can derive the equation of balance of energy (3.8) for a directed curve provided we make the identification

$$\lambda\epsilon = \iint \mu\epsilon^*\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2, \tag{7.16}$$

in addition to (7.8)–(7.13).

8. RESTRICTIONS ON HEAT FLUX VECTORS AND INTERNAL ENERGY

Suppose that the three-dimensional rod-like body is in equilibrium with $\mathbf{v}^* = \mathbf{0}$ and all response functions are independent of the time. As part of their thermodynamic restrictions on constitutive equations, Green and Naghdi[1] have adopted the classical inequality

$$-\mathbf{q}^*\cdot\mathbf{g}^* \geqq 0 \qquad \text{or} \qquad -\mathbf{p}^*\cdot\mathbf{g}^* \geqq 0,$$
$$\mathbf{g}^* = \operatorname{grad}\theta^* \tag{8.1}$$

for all time-independent temperature fields. Integrating the inequality $(8.1)_2$ with respect to ζ^1 and ζ^2, we obtain

$$-\iint g^{1/2}\mathbf{p}^*\cdot\mathbf{g}^*\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2 \geqq 0. \tag{8.2}$$

With the approximation (7.15) for θ^* and with the help of $(7.12)_2$ and (7.13), it follows from (8.2) that

$$\lambda \sum_{\alpha,N} \theta_{\alpha_1\alpha_2\ldots\alpha_N} \bar{\xi}_{\alpha_1\alpha_2\ldots\alpha_N} + k\partial\theta/\partial\zeta + \sum_{\alpha,N} k_{\alpha_1\alpha_2\ldots\alpha_N} \partial\theta_{\alpha_1\alpha_2\ldots\alpha_N}/\partial\zeta \leqq 0 \tag{8.3}$$

for all equilibrium displacement and temperature fields. With the above motivation, we add an inequality of the form (8.3) for all equilibrium states to the thermodynamic inequality (5.4) which was derived directly from one-dimensional postulates.

Next, suppose that the Cosserat curve $\mathcal{R}$ is at rest with

$$\mathbf{v} = \mathbf{0}, \qquad \mathbf{w}_\alpha = \mathbf{0} \tag{8.4}$$

for all time and with the deformation gradient, director and director gradient each constant for all time. Then, by (2.12), ρ is independent of t. In addition, we restrict the temperature fields to be spatially homogeneous so that $\theta = \theta(t)$, $\theta_{\alpha_1\alpha_2\ldots\alpha_N} = 0$ $(N = 1, \ldots, K)$. Keeping these conditions in mind, from a combination of (3.7) and (3.9) we have

$$\lambda\Big(r + \sum_{\alpha,N} r_{\alpha_1\alpha_2\ldots\alpha_N}\Big) - \partial\Big(h + \sum_{\alpha,N} h_{\alpha_1\alpha_2\ldots\alpha_N}\Big)\Big/\partial\zeta = \lambda\dot{\epsilon}. \tag{8.5}$$

In view of (8.4), no mechanical work is supplied to the Cosserat curve $\mathcal{R}$. Then, using (8.5), the heat supplied to a part $\alpha \leqq \zeta \leqq \beta$ of the material curve of $\mathcal{R}$ during the time interval $t_1 \leqq t \leqq t_2$ is

$$\begin{aligned} \mathcal{H} &= \int_{t_1}^{t_2} \left\{ \int_\alpha^\beta \Big(r + \sum_{\alpha,N} r_{\alpha_1\alpha_2\ldots\alpha_N}\Big)\rho \,\mathrm{d}s - \Big[h + \sum_{\alpha,N} h_{\alpha_1\alpha_2\ldots\alpha_N}\Big]_\alpha^\beta \right\} \mathrm{d}t \\ &= \int_\alpha^\beta \rho\epsilon \,\mathrm{d}s \Big|_{t_1}^{t}. \end{aligned} \tag{8.6}$$

Suppose that the rod is in thermal equilibrium during some period up to the time t_1 with constant internal energy ϵ_1 and constant temperature $\theta = \bar{\theta}$. We assume that whenever heat is supplied to the part $\alpha \leqq \zeta \leqq \beta$ under the above conditions, the temperature $\theta(t)$ throughout the part will be increased, i.e.

$$[\theta]_{t_1}^{t_2} > 0 \qquad \text{whenever} \qquad \mathcal{H} > 0. \tag{8.7}$$

Provided that $\rho\epsilon$ is continuous and remembering that ρ, which is independent of t, is positive and α, β are arbitrary, it follows from (8.6) and (8.7) that

$$\theta(t) - \bar{\theta} > 0 \qquad \text{whenever} \qquad \epsilon(t) - \epsilon_1 > 0 \tag{8.8}$$

for all $t > t_1$.

9. SYMMETRIES

In the rest of this paper we restrict attention to the case in which $K = 1$ in the earlier part of the paper (Sections 3, 4 and 5) so that we consider only three temperature fields $\theta, \theta_1, \theta_2$. The inclusion of the two temperatures θ_1, θ_2 allows us to take some account of temperature variations across the thickness of the rod but the more general situation $K > 1$ can be dealt with in a similar way. We also restrict attention to rods which in a reference configuration are straight. We therefore consider the form of the Helmholtz free energy function in (4.13) which is such that the rod described in Section 2 models the main features of a three-dimensional elastic rod which has the following properties in its reference configuration: (i) It is straight with constant cross-sectional areas normal to the line of centroids of sections, (ii) each cross section has two orthogonal axes of symmetry through its center of mass, (iii) it is homogeneous and of constant temperature, (iv) the material of the rod possesses rhombic symmetry with respect to orthogonal axes of symmetry and the line of centroids.

We choose the initial reference curve C to be along the direction of the constant unit vector $\mathbf{e}_3$, and the initial directors in their reference configurations to be specified by

$$\mathbf{D}_1 = D_1\mathbf{e}_1, \qquad \mathbf{D}_2 = D_2\mathbf{e}_2, \tag{9.1}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are a constant orthonormal system of vectors. Also, D_1, D_2 are nonzero constants in line with (i) and (ii) above. The Helmholtz free energy function (4.13) then reduces to the different functional form

$$\psi = \hat{\psi}(\gamma_{ij}, \kappa_{\alpha i}, \theta, \theta_1, \theta_2; D_1, D_2), \tag{9.2}$$

which by virtue of (iii) does not depend explicitly on Θ, ζ. Moreover, recalling (6.5), our theory models a three-dimensional rod-like body which in its reference state has a position vector $\mathbf{R}^*$ specified by

$$\mathbf{R}^* = \zeta\mathbf{e}_3 + \zeta^1 D_1\mathbf{e}_1 + \zeta^2 D_2\mathbf{e}_2. \tag{9.3}$$

Materials with rhombic symmetry with respect to the three directions $\mathbf{e}_i$ contain three classes, namely rhombic-pyramidal, rhombic-disphenoidal and rhombic-dipyramidal.† For each of these classes, it can be shown that the three-dimensional energy function for an elastic material is also form-invariant under the three separate coordinate transformations

$$\text{(a): } \zeta^1 \to -\zeta^1, \qquad \text{(b): } \zeta^2 \to -\zeta^2, \qquad \text{(c): } \zeta \to -\zeta. \tag{9.4}$$

The conditions (9.4) are those which arise from symmetry restrictions usually called orthotropic symmetry. In the direct formulation of the theory of rods under discussion, we replace the notion of rhombic symmetry by that of orthotropic symmetry with respect to three orthogonal vectors $\mathbf{e}_i$. Therefore, we further assume that the one-dimensional energy function (9.2) is unaltered in form under each of the transformations (a)–(c) listed below:

$$\text{(a): } \begin{array}{l} D_1 \to -D_1, \quad D_2 \to D_2, \quad \mathbf{d}_1 \to -\mathbf{d}_1, \quad \mathbf{d}_2 \to \mathbf{d}_2, \\ \theta \to \theta, \quad \theta_1 \to -\theta_1, \quad \theta_2 \to \theta_2, \end{array}$$

i.e. under the transformation

$$\begin{array}{l} G_1 = \{\gamma_{12}, \gamma_{13}, \kappa_{12}, \kappa_{21}, \kappa_{13}, \theta_1, D_1\} \to -G_1, \\ \gamma_{11} \to \gamma_{11}, \quad \gamma_{22} \to \gamma_{22}, \quad \gamma_{33} \to \gamma_{33}, \quad \gamma_{23} \to \gamma_{23}, \\ \kappa_{11} \to \kappa_{11}, \quad \kappa_{22} \to \kappa_{22}, \quad \kappa_{23} \to \kappa_{23}, \quad \theta \to \theta, \quad \theta_2 \to \theta_2, \end{array} \tag{9.5}$$

$$\text{(b): } \begin{array}{l} D_2 \to -D_2, \quad D_1 \to D_1, \quad \mathbf{d}_2 \to -\mathbf{d}_2, \quad \mathbf{d}_1 \to \mathbf{d}_1, \\ \theta \to \theta, \quad \theta_2 \to -\theta_2, \quad \theta_1 \to \theta_1, \end{array}$$

i.e. under the transformation

$$\begin{array}{l} G_2 = \{\gamma_{12}, \gamma_{23}, \kappa_{12}, \kappa_{21}, \kappa_{23}, \theta_2, D_2\} \to G_2, \\ \gamma_{11} \to \gamma_{11}, \quad \gamma_{22} \to \gamma_{22}, \quad \gamma_{33} \to \gamma_{33}, \quad \gamma_{13} \to \gamma_{13}, \\ \kappa_{11} \to \kappa_{11}, \quad \kappa_{22} \to \kappa_{22}, \quad \kappa_{13} \to \kappa_{13}, \quad \theta \to \theta, \quad \theta_1 \to \theta_1 \end{array} \tag{9.6}$$

and

$$\text{(c): } \begin{array}{l} D_1 \to D_1, \quad D_2 \to D_2, \quad \mathbf{D}_3 \to -\mathbf{D}_3, \quad \mathbf{d}_3 \to -\mathbf{d}_3, \quad \mathbf{d}_1 \to \mathbf{d}_1, \quad \mathbf{d}_2 \to \mathbf{d}_2, \\ \theta \to \theta, \quad \theta_1 \to \theta_1, \quad \theta_2 \to \theta_2, \end{array}$$

†See, for example, Green and Adkins[8] which contains additional references on material symmetries.

i.e. under the transformations

$$\begin{aligned}&\gamma_{13}\to-\gamma_{13},\quad \gamma_{23}\to-\gamma_{23},\quad \kappa_{11}\to-\kappa_{11},\quad \kappa_{12}\to-\kappa_{12},\quad \kappa_{21}\to-\kappa_{21},\quad \kappa_{22}\to-\kappa_{22},\\ &\gamma_{11}\to\gamma_{11},\quad \gamma_{22}\to\gamma_{22},\quad \gamma_{33}\to\gamma_{33},\quad \kappa_{13}\to\kappa_{13},\quad \kappa_{23}\to\kappa_{23},\\ &\theta\to\theta,\quad \theta_1\to\theta_1,\quad \theta_2\to\theta_2.\end{aligned} \tag{9.7}$$

Next, in view of the condition (ii), we assume that ψ in (9.2) is unaltered in form under the transformation

$$\text{(d)}:\quad \mathbf{e}_1\to-\mathbf{e}_1,\quad \mathbf{e}_2\to\mathbf{e}_2,\quad \mathbf{e}_3\to\mathbf{e}_3,$$

i.e. under the transformations

$$D_1\to-D_1,\quad D_2\to D_2,\quad \gamma_{ij}\to\gamma_{ij},\quad \kappa_{i\alpha}\to\kappa_{i\alpha},\quad \theta\to\theta,\quad \theta_\alpha\to\theta_\alpha, \tag{9.8a}$$

and ψ is unaltered in form under the transformation

$$\text{(e)}:\quad \mathbf{e}_1\to\mathbf{e}_1,\quad \mathbf{e}_2\to-\mathbf{e}_2,\quad \mathbf{e}_3\to\mathbf{e}_3,$$

i.e. under the transformations

$$D_1\to D_2,\quad D_2\to-D_2,\quad \gamma_{ij}\to\gamma_{ij},\quad \kappa_{i\alpha}\to\kappa_{i\alpha},\quad \theta\to\theta,\quad \theta_\alpha\to\theta_\alpha. \tag{9.8b}$$

A similar discussion may be carried out for the functions ξ, ξ_α, $\bar{\xi}_\alpha$, k, k_α except that each of these functions may depend on $\partial\theta/\partial\zeta$, $\partial\theta_\beta/\partial\zeta$ in addition to those specified in (9.2). In view of the energy identity (3.12), we assume that $\xi\theta$, $\xi_\alpha\theta_\alpha$, $k\partial\theta/\partial\zeta$, $k_\alpha\partial\theta/\partial\zeta$ for $\alpha=1,2$ are each unaltered in form under the transformations (9.5)–(9.8). To each of the transformations (9.5)–(9.8) we add, respectively, the transformations

$$\begin{array}{llll}\text{(a)}: & \partial\theta/\partial\zeta\to\partial\theta/\partial\zeta, & \partial\theta_1/\partial\zeta\to-\partial\theta_1/\partial\zeta, & \partial\theta_2/\partial\zeta\to\partial\theta_2/\partial\zeta\\ \text{(b)}: & \partial\theta/\partial\zeta\to\partial\theta/\partial\zeta, & \partial\theta_1/\partial\zeta\to\partial\theta_1/\partial\zeta, & \partial\theta_2/\partial\zeta\to-\partial\theta_2/\partial\zeta,\\ \text{(c)}: & \partial\theta/\partial\zeta\to-\partial\theta/\partial\zeta. & \partial\theta_\alpha/\partial\zeta\to-\partial\theta_\alpha/\partial\zeta, & \\ \text{(d) and(e)}: & \partial\theta/\partial\zeta\to\partial\theta/\partial\zeta, & \partial\theta_\alpha/\partial\zeta\to\partial\theta_\alpha/\partial\zeta. & \end{array} \tag{9.9}$$

After applying invariance conditions to ψ, ξ, ξ_α, k, k_α using (9.5)–(9.9) we specify the initial directors to be coincident with the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, i.e.

$$D_1=1,\qquad D_2=1. \tag{9.10}$$

In order to make the above conditions explicit we limit our attention to the linear theory of a thermoelastic rod which is in equilibrium, unstressed and at uniform temperature in its reference configuration. For this linear theory the position vector $\mathbf{r}$, directors $\mathbf{d}_\alpha$ and their corresponding velocity fields assume the forms

$$\begin{aligned}&\mathbf{r}=\mathbf{R}+\mathbf{u},\qquad \mathbf{d}_i=\mathbf{D}_i+\bar{\boldsymbol{\delta}}_i,\qquad \mathbf{v}=\dot{\mathbf{u}},\qquad \mathbf{w}_\alpha=\dot{\bar{\boldsymbol{\delta}}}_\alpha,\\ &\mathbf{u}=u_i\mathbf{e}_i,\qquad \bar{\boldsymbol{\delta}}_i=\bar{\delta}_{ij}\mathbf{e}_j,\qquad \mathbf{v}=\dot{u}_i\mathbf{e}_i,\qquad \mathbf{w}_\alpha=\dot{\bar{\delta}}_{\alpha j}\mathbf{e}_j\end{aligned} \tag{9.11}$$

and we replace θ by $\Theta+\theta$ so that θ, θ_1, θ_2 are small and Θ is the constant reference temperature of the rod. Using (9.10), the kinematic measures of deformation become

$$\begin{aligned}&\gamma_{ij}=\mathbf{e}_i\cdot\bar{\boldsymbol{\delta}}_j+\mathbf{e}_j\cdot\bar{\boldsymbol{\delta}}_i=\bar{\delta}_{ij}+\bar{\delta}_{ji},\\ &\kappa_{\alpha i}=\partial\bar{\delta}_{\alpha i}/\partial\zeta,\qquad \bar{\delta}_{3i}=\partial u_i/\partial\zeta.\end{aligned} \tag{9.12}$$

The response functions $\mathbf{n}$, $\boldsymbol{\pi}_\alpha$, $\mathbf{p}_\alpha$ and external fields $\mathbf{f}$, $\mathbf{l}_\alpha$ are referred to the orthonormal basis $\mathbf{e}_i$. Thus

$$\mathbf{n} = n_i\mathbf{e}_i, \qquad \boldsymbol{\pi}_\alpha = \pi_{\alpha i}\mathbf{e}_i, \qquad \mathbf{p}_\alpha = p_{\alpha i}\mathbf{e}_i, \qquad \mathbf{f} = f_i\mathbf{e}_i, \qquad \mathbf{l}_\alpha = l_{\alpha i}\mathbf{e}_i \tag{9.13}$$

and the relations (4.16) simplify to

$$\begin{gathered} n_3 = 2\rho\frac{\partial\hat{\psi}}{\partial\gamma_{33}}, \qquad n_\alpha = \rho\frac{\partial\hat{\psi}}{\partial\gamma_{\alpha 3}}, \\ \pi_{\alpha\beta} + \pi_{\beta\alpha} = 4\rho\frac{\partial\hat{\psi}}{\partial\gamma_{\alpha\beta}}, \qquad p_{\alpha i} = \rho\frac{\partial\hat{\psi}}{\partial\kappa_{\alpha i}}, \\ \eta = -\frac{\partial\hat{\psi}}{\partial\theta}, \qquad \eta_\alpha = -\frac{\partial\hat{\psi}}{\partial\theta_\alpha} \end{gathered} \tag{9.14}$$

where ρ is now the constant reference density and in evaluating $(9.14)_{2,3}$ the response function $\hat{\psi}$ is regarded as a function of $\gamma_{\alpha 3}$ and $(1/2)(\gamma_{\alpha\beta}+\gamma_{\beta\alpha})$. The equations of motion (2.12)–(2.15) are replaced by their corresponding linearized forms in terms of n_i, $\pi_{\alpha i}$ and $p_{\alpha i}$ with $\lambda = \rho$, and the linearized entropy balance equations still have the forms (3.7) with $\lambda = \rho$. All these equations are listed below in four groups.

We omit detailed discussion of the invariance conditions satisfied by ψ, which is now a quadratic form in γ_{ij}, $\kappa_{\alpha i}$, θ, θ_α but note that the final results separate into four groups corresponding to flexure (2 modes), extension and torsion of the rod. Similarly the symmetry restrictions on ξ, ξ_α, k, k_α, which are now linear forms of degree one in γ_{ij}, $\kappa_{\alpha i}$, θ, θ_α, $\partial\theta/\partial\zeta$ and $\partial\theta_\alpha/\partial\zeta$ place these functions in the same four categories. In addition, these functions are subject to the identity (4.7), which in the present context becomes

$$\rho(\bar{\theta}+\theta)\xi + \rho\theta_1\xi_1 + \rho\theta_2\xi_2 + k\partial\theta/\partial\zeta + k_1\partial\theta_1/\partial\zeta + k_2\partial\theta_2/\partial\zeta = 0, \tag{9.15}$$

and the inequality (8.3), namely

$$\rho\theta_1\bar{\xi}_1 + \rho\theta_2\bar{\xi}_2 + k\partial\theta/\partial\zeta + k_1\partial\theta_1/\partial\zeta + k_2\partial\theta_2/\partial\zeta \leqq 0. \tag{9.16}$$

Symmetry restrictions of the type considered in this section must also apply to the kinetic energy (2.5) so that the inertia coefficients $y^{\alpha\beta}$ reduce to

$$y^{11} = \alpha_1, \qquad y^{22} = \alpha_2, \qquad y^{12} = y^{21} = 0, \tag{9.17}$$

where for a homogeneous rod, α_1, α_2 are constants. The Helmholtz free energy function ψ can be expressed in the form†

$$\psi = \psi_{F1} + \psi_{F2} + \psi_E + \psi_T, \tag{9.18}$$

where the subscripts $F1$, $F2$, E, T attached to ψ on the right-hand side of (9.18) refer to the four modes of deformation mentioned above and we recall that we adopt the values (9.10) for D_1, D_2 after the discussion of invariance. The values of the constituent terms in (9.18) are listed below and we also include the notation m_1, m_2, m_3 for the components of the couple $\mathbf{m}$ acting over any section ζ = constant of the rod where

$$\mathbf{m} = m_i\mathbf{e}_i. \tag{9.19}$$

All response functions n_i, $\pi_{\alpha i}$, $p_{\alpha i}$, ξ, ξ_α, k, k_α are assumed to vanish in the reference

†In (9.18), as well as the rest of this section, we use the same symbol for a function and its value.

configuration. The identity (9.15) and the inequality (9.16) are also used in obtaining expressions for ξ, ξ_α, $\bar{\xi}_\alpha$, k, k_α.

*Flexure F*1

$$
\begin{gathered}
2\rho\psi_{F1} = k_5\gamma_{23}^2 + k_{15}\kappa_{23}^2 + k_{25}\kappa_{23}\theta_2 + k_{20}(\theta_2)^2, \\
\pi_{23} = n_2 = k_5\gamma_{23}, \qquad m_1 = p_{23} = k_{15}\kappa_{23} + \frac{1}{2}k_{25}\theta_2, \\
\rho\eta_2 = -k_{20}\theta_2 - \frac{1}{2}k_{25}\kappa_{23}, \qquad \xi_2 = c_1\kappa_{23} + c_2\theta_2, \qquad \bar{\xi}_2 = \bar{c}_2\theta_2, \qquad k_2 = e_2\partial\theta_2/\partial\zeta, \\
\gamma_{23} = \bar{\delta}_{23} + \partial u_2/\partial\zeta, \qquad \kappa_{23} = \partial\bar{\delta}_{23}/\partial\zeta, \\
\partial n_2/\partial\zeta + \rho f_2 = \rho\partial^2 u_2/\partial t^2, \\
\partial m_1/\partial\zeta - n_2 + \rho l_{23} = \rho\alpha_2\partial^2\bar{\delta}_{23}/\partial t^2, \\
\rho\partial\eta_2/\partial t = \rho(s_2 + \xi_2) - \partial k_2/\partial\zeta, \\
\bar{c}_2 \leqq 0, \qquad e_2 \leqq 0.
\end{gathered} \tag{9.20}
$$

*Flexure F*2†

$$
\begin{gathered}
2\rho\psi_{F2} = k_6\gamma_{13}^2 + k_{16}\kappa_{13}^2 + k_{24}\kappa_{13}\theta_1 + k_{19}(\theta_1)^2, \\
\pi_{13} = n_1 = k_6\gamma_{13}, \qquad m_2 = -p_{13} = -k_{16}\kappa_{13} - \frac{1}{2}k_{24}\theta_1, \\
\rho\eta_1 = -k_{19}\theta_1 - \frac{1}{2}k_{24}\kappa_{13}, \qquad \xi_1 = b_1\kappa_{13} + b_2\theta_1, \qquad \bar{\xi}_1 = \bar{b}_2\theta_1, \qquad k_1 = e_1\partial\theta_1/\partial\zeta, \\
\gamma_{13} = \bar{\delta}_{13} + \partial u_1/\partial\zeta, \qquad \kappa_{13} = \partial\bar{\delta}_{13}/\partial\zeta, \\
\partial n_1/\partial\zeta + \rho f_1 = \rho\partial^2 u_1/\partial t^2, \\
\partial m_2/\partial\zeta + n_1 - \rho l_{13} = -\rho\alpha_1\partial^2\bar{\delta}_{13}/\partial t^2, \\
\rho\partial\eta_1/\partial t = \rho(s_1 + \xi_1) - \partial k_1/\partial\zeta, \\
\bar{b}_2 \leqq 0, \qquad e_1 \leqq 0.
\end{gathered} \tag{9.21}
$$

Extension E‡

$$
\begin{gathered}
2\rho\psi_E = \bar{k}_1\gamma_{11}^2 + \bar{k}_2\gamma_{22}^2 + k_3\gamma_{33}^2 + k_7\gamma_{11}\gamma_{22} + k_8\gamma_{11}\gamma_{33} + k_9\gamma_{22}\gamma_{33} \\
+ k_{10}\kappa_{11}^2 + k_{11}\kappa_{22}^2 + k_{17}\kappa_{11}\kappa_{22} + k_{21}\gamma_{11}\theta + k_{22}\gamma_{22}\theta + k_{23}\gamma_{33}\theta + k_{18}(\theta)^2, \\
\pi_{11} = 2\bar{k}_1\gamma_{11} + k_7\gamma_{22} + k_8\gamma_{33} + k_{21}\theta, \qquad \pi_{22} = k_7\gamma_{11} + 2\bar{k}_2\gamma_{22} + k_9\gamma_{33} + k_{22}\theta, \\
n_3 = k_8\gamma_{11} + k_9\gamma_{22} + 2k_3\gamma_{33} + k_{23}\theta, \\
p_{11} = k_{10}\kappa_{11} + \frac{1}{2}k_{17}\kappa_{22}, \qquad p_{22} = \frac{1}{2}k_{17}\kappa_{11} + k_{11}\kappa_{22}, \\
\xi = 0, \qquad \rho\eta = -\frac{1}{2}k_{21}\gamma_{11} - \frac{1}{2}k_{22}\gamma_{22} - \frac{1}{2}k_{23}\gamma_{33} - k_{18}\theta, \qquad k = g_1\partial\theta/\partial\zeta, \\
\gamma_{11} = 2\bar{\delta}_{11}, \qquad \gamma_{22} = 2\bar{\delta}_{22}, \qquad \gamma_{33} = 2\partial u_3/\partial\zeta, \\
\kappa_{11} = \partial\bar{\delta}_{11}/\partial\zeta, \qquad \kappa_{22} = \partial\bar{\delta}_{22}/\partial\zeta, \\
\partial n_3/\partial\zeta + \rho f_3 = \rho\partial^2 u_3/\partial t^2, \\
\partial p_{11}/\partial\zeta + \rho l_{11} - \pi_{11} = \rho\alpha_1\partial^2\bar{\delta}_{11}/\partial t^2, \\
\partial p_{22}/\partial\zeta + \rho l_{22} - \pi_{22} = \rho\alpha_2\partial^2\bar{\delta}_{22}/\partial t^2, \\
\rho\partial\eta/\partial t = \rho s - \partial k/\partial\zeta, \qquad g_1 \leqq 0.
\end{gathered} \tag{9.22}
$$

†This corrects two misprints in eqns $(9.5)_{1,7}$ of Green *et al.* [5].

‡In writing the coefficients $\bar{k}_1$ and $\bar{k}_2$ an overbar is used to avoid confusion with the entropy fluxes k_1, k_2 in $F1$ and $F2$.

Torsion T

$$
\begin{gathered}
2\rho\psi_T = \frac{1}{4}k_4(\gamma_{12}+\gamma_{21})^2 + k_{12}\kappa_{12}^2 + k_{13}\kappa_{21}^2 + k_{14}\kappa_{12}\kappa_{21},\\
\pi_{12} = \pi_{21} = \frac{1}{2}k_4(\gamma_{12}+\gamma_{21}), \qquad p_{12} - p_{21} = m_3,\\
p_{12} = k_{12}\kappa_{12} + \frac{1}{2}k_{14}\kappa_{21}, \qquad p_{21} = k_{13}\kappa_{21} + \frac{1}{2}k_{14}\kappa_{12},\\
\gamma_{12} = \gamma_{21} = \bar{\delta}_{12} + \bar{\delta}_{21}, \qquad \kappa_{12} = \partial\bar{\delta}_{12}/\partial\zeta, \qquad \kappa_{21} = \partial\bar{\delta}_{21}/\partial\zeta,\\
\partial p_{12}/\partial\zeta - \pi_{12} + \rho l_{12} = \rho\alpha_1\partial^2\bar{\delta}_{12}/\partial t^2,\\
\partial p_{21}/\partial\zeta - \pi_{21} + \rho l_{21} = \rho\alpha_2\partial^2\bar{\delta}_{21}/\partial t^2.
\end{gathered} \tag{9.23}
$$

10. STRAIGHT ORTHOTROPIC ROD

We consider the special case in which the direct theory models the main properties of a straight three-dimensional linearly elastic rod whose line of centroids is along the constant unit vector $\mathbf{e}_3$. Each section of the rod is symmetric with respect to two directions specified by constant unit vectors $\mathbf{e}_1, \mathbf{e}_2$. In addition the rod is orthotropic with respect to the constant orthonormal basis $\mathbf{e}_i = (e_1, e_2, e_3)$ and it is homogeneous with constant density ρ^*. The coordinates along the basis $\mathbf{e}_i$ are denoted by $x_i = (x, y, z)$, the constant reference temperature by $\bar{\theta}$ and θ^* is temperature which is zero in the reference state. If $\mathbf{u}^* = u_i^*\mathbf{e}_i$ is the displacement vector and e_{ij} the linear components of the strain, then

$$
e_{ij} = \frac{1}{2}(\mathbf{u}^*_{,i}\cdot\mathbf{e}_j + \mathbf{u}^*_{,j}\cdot\mathbf{e}_i) = \frac{1}{2}(u^*_{i,j} + u^*_{j,i}), \tag{10.1}
$$

where $(\ )_{,i} = \partial(\ )/\partial x_i$. The Helmholtz free energy function ψ^* now takes the form

$$
\rho^*\psi^* = \frac{1}{2}c^{ij}_{kl}e_{ij}e_{kl} - c_{ij}e_{ij}\theta^* - \frac{1}{2}\rho^*(c/\bar{\theta})\theta^{*2} - \rho^*\eta_0\theta^*, \tag{10.2}
$$

where

$$
c_{ij} = c_{ji}, \qquad c^{ij}_{kl} = c^{kl}_{ij} = c^{ji}_{kl} = c^{ij}_{lk}. \tag{10.3}
$$

The coefficients c, η_0, c_{ij}, c^{ij}_{kl} are constants and, since the rod is orthotropic, the only nonzero components are†

$$
c_{11},\quad c_{22},\quad c_{33},\quad c^{11}_{11},\quad c^{11}_{22},\quad c^{11}_{33},\quad c^{22}_{22},\quad c^{22}_{33},\quad c^{33}_{33},\quad c^{23}_{23},\quad c^{13}_{13},\quad c^{12}_{12}. \tag{10.4}
$$

The corresponding components of the stress tensor t_{ij} and the entropy η^* are given by

$$
t_{ij} = c^{kl}_{ij}\,e_{kl} - c_{ij}\theta^*, \qquad \rho^*\eta^* = c_{ij}e_{ij} + \rho^*(c/\bar{\theta})\theta^*. \tag{10.5}
$$

Also, the heat flux vector $\mathbf{q}^*$ and entropy flux vector $\mathbf{p}^*$ have the forms

$$
\mathbf{q}^* = \bar{\theta}\mathbf{p}^*, \qquad \mathbf{p}^* = p_i^*\mathbf{e}_i, \qquad p_i^* = -(d_{ij}/\bar{\theta})\theta^*_{,j}, \tag{10.6}
$$

where $d_{ij} = 0$ if $i \neq k$ and d_{ij} are constants. For an elastic body

$$
\rho\xi^*(\bar{\theta} + \theta^*) = -\mathbf{p}^*\cdot\mathbf{g}^*, \qquad g^* = \text{grad}\,\theta^* \tag{10.7}
$$

and if only linear terms are retained in ξ^*, then

$$
\xi^* = 0. \tag{10.8}
$$

†See Green and Zerna[9, Section 5.4].

The position vector of points in the rod in its reference state is given by

$$\mathbf{R}^* = x_i\mathbf{e}_i = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \tag{10.9}$$

and this corresponds to (9.3) with $x = \zeta^1$, $y = \zeta^2$, $z = \zeta$ and $D_1 = D_2 = 1$. The integrals in (6.6), (6.7), (6.8), (6.9), (7.8)–(7.13) and (7.16) are all with respect to cross-sectional areas in the reference configuration of the rod and their boundaries

$$F(x, y) = 0, \tag{10.10}$$

with $\boldsymbol{\lambda} = \mathbf{e}_3$ and $\lambda^\alpha = 0$ in (6.10). Then, from (6.6) with $\zeta^1 = x$, $\zeta^2 = y$, $\zeta^3 = z$, we have

$$\mu = \rho^*, \qquad \lambda = \rho = A\rho^*, \qquad \lambda\alpha_1 = \lambda y^{11} = \rho^* I_2, \qquad \lambda\alpha_2 = \lambda y^{22} = \rho^* I_1, \tag{10.11}$$

where

$$A = \iint \mathrm{d}x\,\mathrm{d}y, \qquad I_1 = \iint y^2\,\mathrm{d}x\,\mathrm{d}y, \qquad I_2 = \iint x^2\,\mathrm{d}x\,\mathrm{d}y, \tag{10.12}$$

the integrals being over any cross section of the rod.

In order to make identification of the thermoelastic coefficients occurring in the direct linear theory in (9.20)–(9.23) we make some use of the linear results (9.11) and (9.12), together with the approximations (6.5) and (7.15). Thus,

$$\begin{gathered} \mathbf{u}^* = \mathbf{u} + x\bar{\boldsymbol{\delta}}_1 + y\bar{\boldsymbol{\delta}}_2 = (u_i + x\bar{\delta}_{1i} + y\bar{\delta}_{2i})\mathbf{e}_i, \\ \theta^* = \theta + x\theta_1 + y\theta_2, \\ \gamma_{ij} = \bar{\delta}_{ij} + \bar{\delta}_{ji}, \qquad \bar{\delta}_{3i} = \partial u_i/\partial\zeta, \qquad \kappa_{\alpha i} = \partial\bar{\delta}_{\alpha i}/\partial\zeta. \end{gathered} \tag{10.13}$$

We consider first the thermal coefficients in (9.20)–(9.22), some of whose values may be identified by a direct use of the formulae (7.8), (7.11)–(7.13) and the results $(10.5)_2$, (10.6), (10.7) and (10.13). For the coefficients k_{19}, k_{20}, k_{24}, k_{25} we compare simple constant temperature fields θ_1, θ_2 and zero stresses, with the corresponding three-dimensional solutions. Thus

$$F1: \left\{ \begin{aligned} & c_1 = 0, \qquad \bar{c}_2 = c_2 = -Ad_{22}/(\rho\bar{\theta}), \qquad e_2 = -I_1 d_{33}/\bar{\theta}, \\ & k_{20} = -I_1\left[\frac{\rho^* c}{\bar{\theta}} + (c_{11})^2\left(s^{11}_{11} - \frac{(s^{11}_{33})^2}{s^{33}_{33}}\right)\right. \\ & \qquad \left. + 2c_{11}c_{22}\left(s^{11}_{22} - \frac{s^{11}_{33}s^{22}_{33}}{s^{33}_{33}}\right) + (c_{22})^2\left(s^{22}_{22} - \frac{(s^{22}_{33})^2}{s^{33}_{33}}\right)\right], \\ & k_{25} = -2I_1\left[c_{33} + \frac{c_{11}s^{11}_{33} + c_{22}s^{22}_{33}}{s^{33}_{33}}\right], \end{aligned} \right. \tag{10.14}$$

$$F2: \left\{ \begin{aligned} & b_1 = 0, \qquad \bar{b}_2 = b_2 = -Ad_{11}/(\rho\bar{\theta}), \quad e_1 = -I_2 d_{33}/\bar{\theta}, \\ & k_{19} = k_{20}I_2/I_1, \qquad k_{24} = k_{25}I_2/I_1, \end{aligned} \right. \tag{10.15}$$

$$E: \left\{ \begin{aligned} & k_{21} = -Ac_{11}, \qquad k_{22} = -Ac_{22}, \qquad k_{23} = -Ac_{33}, \qquad k_{18} = -\rho^* Ac/\bar{\theta}, \\ & g_1 = -Ad_{33}/\bar{\theta}. \end{aligned} \right. \tag{10.16}$$

where the coefficients s^{rs}_{mn} are defined in (10.18) below.

Turning to the mechanical coefficients $k_1, \ldots, k_{17}$, we note that values for many of these have been given for isotropic rods by Green *et al.*[5]. Here we follow similar procedures but now applied to orthotropic rods which also have two geometrical axes of symmetry in each cross section. We consider first the flexure groups $F1$ and $F2$ in (9.20) and (9.21). A comparison of the solutions of eqns (9.20) and (9.21) for the problem of pure bending of the rod by couples over its ends, with the corresponding solution of the three-dimensional equations suggest that we take

$$F1, F2: \qquad k_{15} = I_1/s^{33}_{33}, \qquad k_{16} = I_2/s^{33}_{33}, \tag{10.17}$$

where s^{ij}_{kl} are the inverse coefficients to c^{ij}_{mn} defined by

$$s^{ij}_{kl} c^{mn}_{ij} = \frac{1}{2}(\delta_k{}^m \delta_l{}^n + \delta_l{}^m \delta_k{}^n), \tag{10.18}$$

where $\delta_j{}^i$ is the Kronecker delta and s^{ij}_{rs} are subject to the same symmetry restrictions as c^{ij}_{rs} in (10.3). When the rod is isotropic the values (10.17) reduce to those given by Green *et al.*[7] and Green *et al.*[10]. In order to specify values for the remaining coefficients k_5, k_6 in $F1$, $F2$, we consider the complete solution of the static isothermal problem in $F2$ in which the rod is unloaded along its major surfaces. Thus, from (9.21), when $f_1 = l_{13} = 0$, the static equations may be integrated in the form

$$\begin{gathered} n_1 = N, \qquad m_2 = M - Nz, \\ \bar{\delta}_{13} = \frac{Nz^2}{2k_{16}} - \frac{Mz}{k_{16}} + R, \qquad u_1 = -\frac{Nz^3}{6k_{16}} + \frac{Mz^2}{2k_{16}} + \left(\frac{N}{k_6} - R\right)z + S, \end{gathered} \tag{10.19}$$

where N, M, R, S are constants. This represents the solution of the flexure problem in which a beam of length l in the region $0 \leqq z \leqq l$ is loaded along an axis of symmetry in the x-direction by a load N. If the couple is zero at $z = l$ and the rod is clamped at the end $z = 0$ so that $u_1 = 0$, $\bar{\delta}_{13} = 0$ there, then

$$m_2 = N(l - z), \qquad \bar{\delta}_{13} = \frac{Nz}{k_{16}}\left(\frac{z}{2} - l\right), \qquad u_1 = \frac{Nz^2}{2k_{16}}\left(l - \frac{z}{3}\right) + \frac{Nz}{k_6}. \tag{10.20}$$

The coefficient k_6 only appears in the expression for the displacement u_1 so that in order to find a suitable value for k_6 we must consider the complete three-dimensional flexure problem including displacements. The flexure of a symmetric orthotropic rod by a force N at the end $z = l$, along an axis of symmetry in the x direction, is specified by displacements

$$\begin{aligned} u^* &= -\frac{N}{2I_2}(s^{11}_{33}x^2 - s^{22}_{33}y^2)(l - z) + \frac{Ns^{33}_{33}}{I_2}\left(\frac{lz^2}{2} - \frac{z^3}{6}\right) + Pz, \\ v^* &= -\frac{Ns^{22}_{33}}{I_2}(l - z)xy, \\ w^* &= -\frac{N}{I_2}\left\{s^{33}_{33}\left(lz - \frac{1}{2}z^2\right)x + \Phi(x, y) + Lxy^2\right\} - Px, \end{aligned} \tag{10.21}$$

where P is a constant representing a rigid body rotation and L is a constant given by

$$\frac{2L}{s^{23}_{23}} = \frac{s^{11}_{33}}{s^{13}_{13}} + \frac{s^{22}_{33}}{s^{23}_{23}} + 4. \tag{10.22}$$

Also,

$$s^{23}_{23}\frac{\partial^2\Phi}{\partial x^2} + s^{13}_{13}\frac{\partial^2\Phi}{\partial y^2} = 0 \tag{10.23}$$

subject to the boundary condition

$$l\left\{\frac{\partial\Phi}{\partial x} + \left(L + \frac{1}{2}s^{22}_{33}\right)y^2 - \frac{1}{2}s^{11}_{33}x^2\right\}s^{23}_{23} + m\left\{\frac{\partial\Phi}{\partial y} + (2L - s^{22}_{33})xy\right\}s^{13}_{13} = 0 \tag{10.24}$$

over the surface (10.10), where (l, m) are direction cosines of the outward normal to this surface. Recalling the approximation (10.13) for $\mathbf{u}^*$, from the comparison of $(10.21)_{1,3}$ with (10.20) we again obtain the identification $(10.17)_2$ for k_{16}. In view of $(10.20)_2$ we choose the rigid rotation P in $(10.21)_3$, and the function Φ, which is even in y and odd in x and is arbitrary to the extent of an additive constant, so that

$$\Phi(0,0)=0, \qquad P=-N\frac{\partial\Phi(0,0)}{\partial x}\Big/ I_2. \tag{10.25}$$

Then, comparison of $(10.20)_3$ and (10.21) at the end $z=0$ of the rod for small values of x, y yields $P = N/k_6$ so that

$$F2: \qquad k_6=-I_2\Big/\frac{\partial\Phi(0,0)}{\partial x}. \tag{10.26}$$

Analytical solutions for the flexure problem specified by (10.23) and the surface condition (10.24) may be found for a number of simple cross sections of the rod. We merely quote the final value obtained for k_6 for a circular cross section of radius R, namely

$$F2: \qquad k_6=\frac{\pi R^2(3s_{13}^{13}+s_{23}^{23})}{4s_{13}^{13}(4s_{13}^{13}+2s_{23}^{23}+s_{33}^{22})}. \tag{10.27}$$

Similarly, when the section is circular, for flexure $F1$ we have

$$F1: \qquad k_5=\frac{\pi R^2(3s_{23}^{23}+s_{13}^{13})}{4s_{23}^{23}(4s_{23}^{23}+2s_{13}^{13}+s_{33}^{11})}. \tag{10.28}$$

When the rod is isotropic, these reduce to the common value

$$k_5=k_6=2\mu\pi R^2(1+\nu)/(3+2\nu), \tag{10.29}$$

where ν is the Poisson ratio. This may be compared with a value

$$6\mu\pi R^2(1+\nu)^2/(7+14\nu+8\nu^2)$$

found by Green *et al.*[5] by a different procedure which is slightly greater than (10.29) for values of ν in the range $0\leqq\nu\leqq 1/2$.

For the torsion group T we compare first the solution of eqn (9.23) with the Saint-Venant torsion problem in linear three-dimensional theory in the manner described by Green *et al.*[10]. In the present context, for an orthotropic rod, this leads to the choice

$$T: \qquad k_{12}=k_{13}, \qquad 2k_{12}-k_{14}=\mathscr{D}, \tag{10.30}$$

where $\mathscr{D}$ is the classical torsional rigidity for the rod. With the choice (10.30), the isothermal static equations in (9.23) divide themselves into two groups, namely

$$m_3=\left(k_{12}-\frac{1}{2}k_{14}\right)\partial(\bar{\delta}_{12}-\bar{\delta}_{21})/\partial z, \qquad \partial m_3/\partial z+\rho(l_{12}-l_{21})=0 \tag{10.31}$$

and

$$\begin{gathered} p_{12}+p_{21}=\left(k_{12}+\frac{1}{2}k_{14}\right)\partial(\bar{\delta}_{12}+\bar{\delta}_{21})/\partial z, \qquad \pi_{12}=\pi_{21}=k_4(\bar{\delta}_{12}+\bar{\delta}_{21}), \\ [\partial(p_{12}+p_{21})/\partial z]-2\pi_{12}+\rho(l_{12}+l_{21})=0. \end{gathered} \tag{10.32}$$

We now consider the three-dimensional solution for dispalcements and stresses given by

$$\begin{gathered} u_1^*=Lyz, \qquad u_2^*=Lxz, \qquad u_3^*=0, \\ t_{12}=2c_{12}^{12}Lz, \qquad t_{13}=Lc_{13}^{13}y, \qquad t_{23}=Lc_{23}^{23}x. \end{gathered} \tag{10.33}$$

The system of stresses in (10.33) is in equilibrium and can be maintained with suitable surface tractions. Corresponding to this we see from (10.13) that an exact solution of eqns (10.31) and (10.32) is

$$\bar{\delta}_{12} = \bar{\delta}_{21} = Lz, \qquad \rho l_{12} = \rho l_{21} = \pi_{12} = \pi_{21},$$
$$p_{12} + p_{21} = 2\left(k_{12} + \frac{1}{2}k_{14}\right)L, \qquad \pi_{12} = \pi_{21} = 2k_4 Lz. \tag{10.34}$$

With the help of (6.7), (6.8) and (10.33), we make the identifications

$$p_{12} + p_{21} = (I_1 c^{13}_{13} + I_2 c^{23}_{23})L, \qquad \pi_{12} = 2Ac^{12}_{12}Lz. \tag{10.35}$$

This leads us to choose

$$T: \quad k_{12} + \frac{1}{2}k_{14} = \frac{1}{2}(I_1 c^{13}_{13} + I_2 c^{23}_{23}), \qquad k_4 = Ac^{12}_{12}. \tag{10.36}$$

A complete set of torsional coefficients is given by (10.30) and (10.36).

Finally we consider the extension problem governed by eqns (9.22). Following Green *et al.* [5], the values of many of the coefficients can be specified by a comparison of static solutions of (9.22) with homogeneous deformations of the three-dimensional rod. This yields the values

$$E: \quad \begin{array}{lll} k_1 = \frac{1}{4}Ac^{11}_{11}, & k_2 = \frac{1}{4}Ac^{22}_{22}, & k_3 = \frac{1}{4}Ac^{33}_{33}, \\ k_7 = \frac{1}{2}Ac^{11}_{22}, & k_8 = \frac{1}{2}Ac^{11}_{33}, & k_9 = \frac{1}{2}Ac^{22}_{33}. \end{array} \tag{10.37}$$

When the rod is isotropic, the results (10.37) reduce to those given in Green *et al.* [5]. Next, we consider the three-dimensional solution for displacements and stresses given by

$$\begin{array}{ll} u_1^* = Lxz, \quad u_2^* = Myz, & u_3^* = \frac{1}{2}Nz^2, \\ t_{11} = (c^{11}_{11}L + c^{11}_{22}M + c^{11}_{33}N)z, & t_{12} = 0, \\ t_{22} = (c^{11}_{22}L + c^{22}_{22}M + c^{22}_{33}N)z, & t_{13} = c^{13}_{13}Lx, \\ t_{33} = (c^{11}_{33}L + c^{22}_{33}M + c^{33}_{33}N)z, & t_{23} = c^{23}_{23}My, \end{array} \tag{10.38}$$

where L, M, N are constants. The system of stresses in (10.38) satisfies the equations of equilibrium if

$$(c^{13}_{13} + c^{11}_{33})L + (c^{23}_{23} + c^{22}_{33})M + c^{33}_{33}N = 0, \tag{10.39}$$

and can be maintained by suitable surface tractions. Recalling the expressions (10.13), it follows that the corresponding results in the direct extensional theory (9.22) are:

$$\begin{gathered} \gamma_{11} = 2\bar{\delta}_{11} = 2Lz, \qquad \gamma_{22} = 2\bar{\delta}_{22} = 2Mz, \qquad \gamma_{33} = 2\partial u_3/\partial z = 2Nz, \\ \kappa_{11} = L, \qquad \kappa_{22} = M, \\ \rho l_{11} = \pi_{11} = A(c^{11}_{11}L + c^{11}_{22}M + c^{11}_{33}N)z, \\ \rho l_{22} = \pi_{22} = A(c^{11}_{22}L + c^{22}_{22}M + c^{22}_{33}N)z, \\ \rho f_3 + A(c^{11}_{33}L + c^{22}_{33}M + c^{33}_{33}N) = 0, \\ p_{11} = k_{10}L + \frac{1}{2}k_{17}M, \qquad p_{22} = \frac{1}{2}k_{17}L + k_{11}M. \end{gathered} \tag{10.40}$$

With the help of (6.7), (6.8) and (10.38) we make the identification

$$p_{11} = I_2 c^{13}_{13}L, \qquad p_{22} = I_1 c^{23}_{23}M. \tag{10.41}$$

Comparison of (10.41) with the corresponding expressions in (10.40) leads us to choose

$$E: \qquad k_{10} = I_2 c^{13}_{13}, \qquad k_{11} = I_1 c^{23}_{23}, \qquad k_{17} = 0 \tag{10.42}$$

for the values of k_{10}, k_{11}, k_{17}. A complete set of values of coefficients for the extension problem are contained in (10.16), (10.37) and (10.42).

When the rod is isotropic and has circular cross sections of radius R we see from (10.30), (10.36), (10.37) and (10.42) that

$$\begin{gathered} k_{10} = k_{11} = k_{12} = k_{13} = \frac{1}{4}\mu\pi R^4, \qquad k_{14} = k_{17} = 0, \\ k_4 = 2k_1 - k_7 = \mu\pi R^2. \end{gathered} \tag{10.43}$$

Previously (Green *et al.*[10]), all constitutive coefficients in the case of an isotropic rod of circular cross section were determined except k_{10}, k_{11} and k_{17}. The above more general development, which is valid for orthotropic rods of circular cross section, upon specialization also provides appropriate values for the coefficients k_{10}, k_{11} and k_{17} in the isotropic case.

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